Terrain inclination and curvature from wavelet coefficients
Approximation formulae for the relief

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Abstract. The wavelet transformation (WT) of a function \( y = f(x) \) with a wavelet of the \( n \)th order approximates the \( n \)th derivative of the function except for a constant scaling factor and a frequency-dependent phase shift. In the two-dimensional WT, the equivalent applies to the partial derivatives of a function \( z = f(x, y) \). If a digital terrain model (DTM) has been stored in form of wavelet coefficients (WCs), then the gradient and, if required, also curvature values can be directly deduced from the WCs. By means of special functions (test functions) whose derivatives are known, the scaling (‘amplitude correction’) and the displacement (‘phase correction in the space domain’) can be determined. The moments of the wavelets and the scaling functions (high- and low-pass filters) make it possible to derive the approximation formulae in a clear and wavelet-independent manner.

Keywords: Wavelet transformation – Digital terrain model – Terrain inclination – Differential geometry

1 Introduction

Wavelet transformation (WT) has proved to be a suitable tool for many problems related to the analysis and processing of signals with complex structures. This is due to special features of WT, such as the localization property, data compression, filtering in the wavelet domain (WD), and the calculation speed of the discrete realization with regular data (multiresolution analysis – MRA).

Another excellent feature – in textbooks and monographs only mentioned in passing, if at all – is the so-called approximation possibility, which allows us, under certain conditions, to derive the first, second and higher derivatives of a signal directly from its wavelet transform. Among other things, edge detection in pictures is based on this feature (see Mallat and Zhong 1992).

For many applications in the geosciences the terrain inclination is an essential quantity (e.g. for drainage areas, inundation forecast, erodible and avalanche-prone slopes etc.); sometimes the curvature is also needed (see Bethge 1994). If a digital terrain model (DTM) has been stored in the form of wavelet coefficients (WCs), then the gradient and, if required, also curvature values can be directly deduced from the WCs without the need to execute a back-transformation into the original domain and numerically differentiate there.

The wavelet transform of a signal, multiplied by a scale-dependent factor, approximates the \( n \)th derivative of the signal by a limiting process for the scale parameter \( a \to 0 \) provided that the used wavelet is of the \( n \)th order.

A wavelet is understood to be of the \( n \)th order if its mean and first \( n-1 \) moments vanish while the \( n \)th moment is finite and unequal to zero. The order of the wavelet controls the behaviour of the WT for small \( a \).

In the monograph of Louis et al. (1994), this so-called high-frequency behaviour was treated with functional analytic methods, including distributive derivatives. The latter can be of interest if we are considering hybrid DTMs with edge information.

During discrete approximation the limiting process \( a \to 0 \) is not possible. The wavelets on even the finest scale have a support \( > 1 \). Hence, the approximation of the derivative represented by a WC is a ‘mean’ of several neighbouring discrete signal values. Thus, the question is whether the discrete WT has an exact localization property too. It turns out that the property of localization depends significantly upon the moments of the wavelets considered.

It follows that two problems are to be solved initially:

1. scaling of the WCs so that they have the value(s) of the sought ‘mean’ derivative(s) (‘amplitude correction’);
2. assignment of the scaled WCs to discrete arguments (‘phase correction’: displacement in the space domain, phase shift in the Fourier domain).
In Beyer and Meier (2001) the theoretical fundamentals were described for the one-dimensional (1-D) case and useful calculation formulae deduced for the inclination and curvature in the 1-D case or in terrain profiles, respectively. This article presents the results for the 2-D case. First, for preparatory purposes, the most essential fundamentals of the usual algorithms of 2-D discrete WT will be listed (Sect. 2). These include the computation of the wavelets by iteration, the construction of 2-D wavelets from 1-D wavelets and the description of the existing relationships of indices. Due to the relationships between the 1- and 2-D WT some statements concerning the 1-D case can directly be transferred to the 2-D case.

In Sect. 3 the necessary scaling factor (Sect. 3.3) and the space lag, i.e. the correction of the localization of the WCS in the space domain, are deduced (Sect. 3.4). In Sect. 4 a synthetic example is given to demonstrate the applicability of the approximation formulae. It will be shown that via the approximated derivatives a wavelet-transformed terrain model also delivers the inclination and curvature of the terrain.

2 Two-dimensional discrete WT

2.1 Tensor product wavelets

2.1.1 Wavelets and scaling functions (high and low pass) Basic concepts and denotations. This list is limited to the connections which are necessary for the following derivations. In particular, the index relationships are important for the localization effects that occur. For the detailed theoretical background the literature should be referred to (Wickerhauser 1994; Strang and Nguyen 1997).

The terrain surface (relief) given in the form of a matrix of heights is a 2-D discrete signal from the point of view of signal processing, or a 2-D discrete function from the point of view of mathematics. In this article both concepts are used synonymously in some cases.

The discrete wavelet transformation is equivalent to a continuous projection of a discrete signal as an element of a function space into a sequence of subspaces \( \mathcal{V}_0 \), their orthogonal complements \( \mathcal{W}_0 \). The basis functions of the subspaces \( \mathcal{V}_0 \) are developed by dilation and translation of a scaling function \( \phi \) (the so-called father function). From this principle the scaling equation follows as

\[
\phi(x) = \sum c_n \cdot \phi(2x - n), \quad n \in \mathbb{Z}
\]  

(1)

with the coefficients \( c_n \) as a necessary condition for the scaling function.

The basis functions of the orthogonal complements \( \mathcal{W}_0 \) are the wavelets (so-called mother functions). From given coefficients \( c_n \), which satisfy the scaling function, the wavelets for the different scales can be determined by iteration (see Strang and Nguyen 1997). We start in the continuous case from a basis \( \{ \phi_{0n} \} \) of a function space \( \mathcal{V}_0 \) with

\[
\tilde{\phi}_{0n}(x) = \begin{cases} 
1 & \text{for } n \Delta x \leq x < (n + 1)\Delta x \\
0 & \text{else}
\end{cases}
\]

This function space corresponds to a discretization with the sampling distance \( \Delta x \). For the examined algorithms of the discrete WT the canonical basis (standard basis) \( \{ \phi_{0n} \} \) with

\[
\phi_{0n} = \begin{bmatrix} \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \end{bmatrix}^T
\]

\( n \in \mathbb{Z}, \phi_{0n} \in \mathcal{V}_0 \)

with the so-called prototype \( \phi_{00} \) takes over the role of the above-mentioned basis \( \{ \phi_{kn} \} \). The prototypes of the father function \( \phi_{k0} \) and the mother function \( \psi_{k0} \) (wavelet) of the \( k \)th scale are produced by iteration

\[
\phi_{k0} = \frac{1}{\sqrt{2}} \sum_n c_n \cdot \phi_{k-1,n}
\]

\[
\psi_{k0} = \frac{1}{\sqrt{2}} \sum_n (-1)^n c_{1-n} \cdot \phi_{k-1,n}
\]

(2)

The factor \( 1/2\sqrt{2} \) guarantees \( ||\phi_{kn}|| = ||\psi_{kn}|| = 1 \). The complete basis of the \( k \)th scale follows by continuous displacement of the prototype with an increment of \( 2^k \). Similarly to the 1-D case, let us describe the finite discrete-given signal \( z_{ij} = f(x_i, y_j) \) by the matrix of values (signal matrix)

\[
Z = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1m} \\
z_{21} & z_{22} & \cdots & z_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
z_{m1} & z_{m2} & \cdots & z_{mm} 
\end{bmatrix}
\]

In order to construct the 2-D wavelets, both the 1-D wavelet \( \psi \) (high pass) and the 1-D scaling function \( \phi \) (low pass) are required. In the interest of compatibility with the usual denotation of the 2-D tensor product wavelets they should be described with \( h \) and \( l \) as a high-pass filter vector

\[
h := \psi = \begin{bmatrix} v_1^h & v_2^h & \cdots & v_s^h \end{bmatrix}^T
\]

and a low-pass filter vector

\[
l := \phi = \begin{bmatrix} v_1^l & v_2^l & \cdots & v_s^l \end{bmatrix}^T
\]

where \( (m,m) \) is the format of the signal matrix and \( s \) is the length (support) of the wavelets at the scale considered. The support of the wavelet is understood to be the domain from the first to the last element unequal to zero of the discrete wavelet, or the length of this domain. The wavelets constructed above are really infinitely long sequences which, limited to their support in this manner, are described as finite vectors.

Example 1. Daubechies4 wavelet. The coefficients of the Daubechies4 wavelet are \( c_0 = \frac{1}{4}(1 + \sqrt{3}) \), \( c_1 = \frac{1}{4}(3 + \sqrt{3}) \), \( c_2 = \frac{1}{4}(3 - \sqrt{3}) \), \( c_3 = \frac{1}{4}(1 - \sqrt{3}) \). We obtain for the high-pass vector \( h \) and the low-pass vector \( l \) at the first scale, as result of Eq. (2)