Decomposition of deformation primitives of horizontal geodetic networks: application to Taiwan’s GPS network

R. Hsu, S. Li

Department of Civil Engineering, National Taiwan University, Taipei 106, Taiwan
e-mail: rshsu@ce.ntu.edu.tw; Tel.: +88-62-2367-9600; Fax: +88-62-23631558

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Abstract. The contribution to each component of the deformation vector due to a marginally undetected blunder in an observation is split into two parts, and so are the deformation measures at a point. The first part, called the local component (or self component), signifies the contribution due to the redundancy-related multiplier belonging to the observation itself, while the second part, called the complementary component, shows the contribution due to the non-redundancy-related multipliers belonging to the observation. A larger redundancy results in a smaller local component for each component of the deformation vector. However, robustness due to an observation at a point is correlated with redundancy to a significant extent. The first-order global positioning system (GPS)-geodetic network of Taiwan was examined for its robustness. The experiments seem to indicate that: (1) large deformations tend to be found at points where the group redundancies are small; (2) the local components monopolize deformation measures at the perimeter stations of the network where very small redundancy numbers are found; (3) the largest deformation at a point may be due to an observation not directly tied to the point of interest; and (4) except for local twisting, deformation measures and mean positional precisions at individual points are highly correlated.

Key words. Geodetic network – Deformation – Reliability – Robustness – Redundancy

1 Introduction

It is well known that blunders in geodetic, or other, observations are likely to go undetected. Traditionally, the influences of maximum undetectable blunders were investigated using the concept of reliability (Baarda 1968). The results of Baarda’s investigations are summarized by internal reliability and external reliability. In the realm of internal reliability, the marginally detectable blunders of individual observations measure the capability of the network to detect blunders. The external reliability of a network, also computed for individual observations, reveals the maximum effects of the undetected blunders on the estimated parameters. The effects due to undetected blunders on the positions of the individual network points have never been explored in the theory of reliability. Since a good internal reliability does not guarantee reliable positions of the network points, we can ask the question: how much can undetected blunders affect the estimated positions? This question has been answered satisfactorily in the theory of robustness analysis (Vaníček et al. 1991, 2001). A network is said to be robust if the influence on the estimated positions due to the undetected blunders is slight. Conversely, if the influence is significant, the network has weak robustness.

According to Vaníček et al. (1991, 2001), the robustness at a point is measured by three deformation primitives: mean strain, total shear, and local differential rotation. Every marginally undetected blunder belonging to an observation gives rise to three deformation measures, and the largest ones are chosen at each point to indicate its robustness in strain, shear, and rotation. Seemkooei (2001a) reveals that “the robustness parameters were affected by redundancy numbers. The largest robustness parameters were due to the observations with minimum redundancy numbers”. It is therefore obvious from Seemkooei’s work that reliability and robustness are closely related. We thus need mathematical expressions that show such functional relationships. However, the expressions of the deformation primitives in Vaníček et al. (1991, 2001) give no clue as to the relationships between the primitives and the redundancy numbers belonging to the observations constituting the network.

In this paper, we endeavor to express the three deformation measures in terms of the redundancy and the marginally undetected blunder. The expressions presented enable us to separate a deformation measure...
due to an observation, say $k$, into two components. The first component, called the local component or self component, shows the contribution due to the redundancy-related multiplier belonging to the $k$th observation itself. The second component, the complementary component, shows the contributions due to non-redundancy-related multipliers belonging to the observation $k$. As an example, the first-order global positioning system (GPS)-geodetic network of Taiwan was explored for its robustness using the formulas derived herein. In particular, the role that the group redundancy plays and the percentage that the local components share in the determinations of deformation measures were studied. Research was also undertaken to examine the correlations between the deformation measures and mean positional precisions.

2 Deformation measures

Blunders in geodetic observations cause displacements at the individual points of a geodetic network, thereby inducing deformation. It is therefore straightforward to measure the degree of robustness by means of network deformation.

Let the two-dimensional (2-D) displacements of a point $p_i(x_i, y_i)$ be

$$
\begin{bmatrix}
\delta x_i \\
\delta y_i
end{bmatrix} = \begin{bmatrix}
u_i \\
v_i
end{bmatrix}
$$

then the deformation matrix at point $p_i$ is defined by the gradient with respect to position, namely (Vaniček et al. 1991, 2001)

$$
E_i = \text{grad} \left( \begin{bmatrix}
\delta x_i \\
\delta y_i
end{bmatrix} \right) = \begin{bmatrix}
\partial u_i/\partial x & \partial u_i/\partial y \\
\partial v_i/\partial x & \partial v_i/\partial y
end{bmatrix}
$$

From the matrix $E_i$, three deformation measures (or primitives) are used at point $p_i$ (Vaniček et al. 1991, 2001); these are as follows.

2.1 Mean strain

$$
\sigma = \frac{1}{2} (\partial u/\partial x + \partial v/\partial y)
$$

which describes the average contraction or extension at a point, and therefore can be regarded as a deformation in scale.

2.2 Total shear

$$
\gamma = \frac{1}{2} \sqrt{\tau^2 + v^2}
$$

which is the geometric mean of pure shear $\tau = \frac{1}{2} (\partial u/\partial x - \partial v/\partial y)$ and simple shear $v = \frac{1}{2} (\partial u/\partial y + \partial v/\partial x)$. Pure shear spoils the separation between two lines; simple shear deforms the angle between two lines. Thus, the total shear reveals the deformation in a local configuration.

2.3 Local twisting

The differential rotation at the point of interest is described by

$$
\omega = \frac{1}{2} (\partial v/\partial x - \partial u/\partial y)
$$

This rotation can be further separated into two components — the block rotation $\omega_o$ and the local differential rotation $\delta \omega$. The former is common to the whole network and computed by

$$
\omega_o \approx \frac{1}{m} \sum \omega_i
$$

where $m$ denotes the number of deformed points. The local rotation at each point is

$$
\delta \omega_i \approx \omega_i - \omega_o
$$

which is used to describe the local twisting.

Thus, the robustness at a point is characterized by these three deformation measures — namely, robustness in scale, robustness in shape and robustness in twist.

3. Evaluation of the deformation matrix

Consider the point $p_i$ and its adjacent points $p_j (j = 1, \ldots, t)$. These $t$ adjacent points are either all points connected by observations to the point $p_i$ or all points within a specified radius from the point of interest. The displacement field of these $(t+1)$ points can be approximated by two planar equations (Vaniček et al. 2001)

$$
\begin{align*}
\mathbf{u}_i &= a_i \mathbf{1} + \frac{\partial u_i}{\partial x} \Delta \mathbf{X}_i + \frac{\partial u_i}{\partial y} \Delta \mathbf{Y}_i \\
\mathbf{v}_i &= b_i \mathbf{1} + \frac{\partial v_i}{\partial x} \Delta \mathbf{X}_i + \frac{\partial v_i}{\partial y} \Delta \mathbf{Y}_i
end{align*}
$$

where $a_i$ and $b_i$ are absolute terms, $\mathbf{u}_i$ and $\mathbf{v}_i$ are the vectors consisting of the displacement components of these $(t+1)$ points, $\mathbf{1}$ is a column vector having ones as components, and $\Delta \mathbf{X}_i$ and $\Delta \mathbf{Y}_i$ are the vectors consisting of the $x$- and $y$-coordinate components, respectively, expressed relative to the point of interest. Solving by least squares (LS) for the unknown partial derivatives and absolute terms in Eq. (8) yields

$$
\begin{bmatrix}
a_i \\
\frac{\partial u_i}{\partial x} \\
\frac{\partial u_i}{\partial y}
end{bmatrix} = (K_i^T K_i)^{-1} K_i^T \mathbf{u}_i
$$

and

$$
\begin{bmatrix}
a_i \\
\frac{\partial v_i}{\partial x} \\
\frac{\partial v_i}{\partial y}
end{bmatrix} = (K_i^T K_i)^{-1} K_i^T \mathbf{v}_i
$$

(9a)