A comparison between the Airy/Heiskanen and the Pratt/Hayford isostatic models for the computation of potential harmonic coefficients

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1 Introduction

There are many brilliant contributions to spherical harmonic analysis of a function given on a sphere as well as to spherical harmonic synthesis which is the forward computation of retrieving the function from its Fourier coefficients (see e.g. Colombo 1981; Sneeuw 1994; Sneeuw and Bun 1996). An important application of this theory is when the function is a global terrain model. The theory of reduction of a global elevation model to spherical harmonic coefficients taking isostasy into consideration can be found in many bibliographic sources (see e.g. Balmino et al. 1973; Rapp 1982; Sünkel 1985, 1986; Rümmel et al. 1988; Pavlis and Rapp 1990). All of these contributions consider the Airy/Heiskanen isostatic model, apart from Sünkel who introduced a smoothing operator to the linearized Vening Meinesz model and determined both depth to the compensation level and the smoothing factor to account for a regional compensation. The scope of the present paper is to revisit the theory of spherical harmonic analysis of a global Digital Elevation Model (DEM) using the Airy/Heiskanen model and expand it for the Pratt/Hayford model as well. The spectra resulting from both models will be computed and compared to the observed gravity field of EGM96.

2 Expansion for the potential of mass distributions into spherical harmonics

For the inverse distance function the following series expression in spherical coordinates holds, or equally its spherical harmonic expansion

\[ \frac{1}{l_{PQ}} = \frac{1}{r_p} \sum_{l=0}^{\infty} \left( \frac{r_p}{r_P} \right)^l P_l(\cos \theta_P) \quad \text{for } r_Q < r_P \]  

(1)

\[ \frac{1}{l_{PQ}} = \frac{1}{r_Q} \sum_{l=0}^{\infty} \left( \frac{r_P}{r_Q} \right)^l P_l(\cos \theta_Q) \quad \text{for } r_P < r_Q \]  

(2)

where \( P_l(\cos \psi_{PQ}) \) are the Legendre polynomials of degree \( l \) and \( \psi_{PQ} \) the angle linking attracting point \( Q \) to the computation point \( P \). A separation of the functions related to \( P \) from those related to \( Q \) can be made by means of the addition theorem of the spherical harmonic functions

\[ P_l(\cos \psi_{PQ}) = \frac{1}{2l+1} \sum_{m=0}^{l} \tilde{P}_l^m(\cos \theta_P) \tilde{P}_l^m(\cos \theta_Q) \times \left( \cos m\lambda_P \cos m\lambda_Q + \sin m\lambda_P \sin m\lambda_Q \right) \]  

(3)

where \( \tilde{P}_l^m \) are the fully normalized associated Legendre functions and \( m \) denotes order. Equation (3) refers to normalized quantities and can be derived from the non-normalized expression given by Lense (1954, pp. 75–76). Using the abbreviation

\[ Y_{l, m}^\pi(P) = \tilde{P}_l^m(\cos \theta_P) \begin{cases} \cos m\lambda_P & \text{for } \pi = 0 \\ \sin m\lambda_P & \text{for } \pi = 1 \end{cases} \]  

(4)

Eq. (3) becomes

\[ P_l(\cos \psi_{PQ}) = \frac{1}{2l+1} \sum_{m=0}^{l} \sum_{\pi=0}^{1} Y_{l, m}^\pi(P) Y_{l, m}^\pi(Q) \]  

(5)

The gravitational potential at an arbitrary point in space \( P \) due to the Earth’s mass distribution is given by Newton’s law of gravitation

\[ V_P = G \int \int \frac{\rho(Q)}{l_{PQ}} d\Sigma_Q \]  

(6)

where \( G \) denotes the gravitational constant, \( \rho \) the density inside the Earth and \( l_{PQ} \) the distance between \( P \) and the infinitesimal volume element \( d\Sigma_Q \) at \( Q \). Inserting Eqs. (1) and (5) into Eq. (6), we obtain
\[ V_p = G \left( \sum_{m=0}^{l} \sum_{n=0}^{l} \left( \frac{r_Q}{r_p} \right)^m \right) \frac{1}{2l+1} \]
\[ \times \int \int \! \! \int \rho(Q) Y_{lm}^2(Q) \, d\Sigma_Q \]
\[ = G \left( \sum_{m=0}^{l} \sum_{n=0}^{l} \left( \frac{r_Q}{r_p} \right)^m \frac{1}{2l+1} \right) \int \int \! \! \int \rho(Q) Y_{lm}^2(Q) \, d\Sigma_Q \]
\[ = \frac{GM}{R} \int \int \! \! \int \rho(Q) Y_{lm}^2(Q) \, d\Sigma_Q \]
\[ = \frac{GM}{R} \int \int \! \! \int \rho(Q) Y_{lm}^2(Q) \, d\Sigma_Q \]
\[ = \left( \frac{R}{R-D_A} \right)^2 \frac{\rho_{\text{cr}}}{\Delta \rho} h \]
\[ t = \frac{\rho_{\text{cr}}}{\Delta \rho} h \]
\[ t' = \left( \frac{R}{R-D_A} \right)^2 \frac{\rho_{\text{cr}} - \rho_w}{\Delta \rho} h' \]

Where \( h \) and \( h' \) denote the positive (heights) and negative elevations (depths) of a global elevation set and \( \rho_w = 1030 \, \text{kg} \, \text{m}^{-3} \) the density of sea water. When the convergence of the verticals is taken into account one obtains in linear approximation (Lambeck 1988; Rummel et al. 1988)

\[ t = \left( \frac{R}{R-D_A} \right)^2 \frac{\rho_{\text{cr}}}{\Delta \rho} h \]

\[ t' = \left( \frac{R}{R-D_A} \right)^2 \frac{\rho_{\text{cr}} - \rho_w}{\Delta \rho} h' \]

R denotes a mean Earth radius value \( (R = 6370 \, \text{km}) \) and \( D_A \) the thickness of the crust for zero elevation. A popular value for \( D_A \) in Airy’s model is \( D_A = 30 \, \text{km} \).

Equation (8) is written in the Airy/Heiskanen model as the difference between the coefficients generated by the potential of the surface topography and those generated by the compensation part. One writes

\[ C_{lm}^T = \frac{3}{\rho R(2l+1)4\pi} \int A^T(Q) \, d\Sigma_Q \]

where the surface topography part is

\[ A^T(Q) = \rho_{\text{cr}} \int_{r=R-h}^{R+h} \left( \frac{r_Q}{R} \right)^{1/2} \, dr_Q \]

and the compensation part

\[ A^C(Q) = \Delta \rho \int_{r=R-D_A-t}^{R-D_A} \left( \frac{r_Q}{R} \right)^{1/2} \, dr_Q \]

Repeating ocean depths \( h' \) by equivalent rock topography and taking the convergence of the verticals into account, one obtains for the coefficients of the isostatically compensated topography (for derivations see Sünkel 1986; Rummel et al. 1988)

\[ C_{lm}^T = C_{lm}^T - C_{lm}^C \]

where the coefficients from the uncompensated topography are

3 Topographic/isostatic harmonic coefficients with the Airy/Heiskanen model

The crust in the Airy/Heiskanen model is considered to have constant density \( \rho_{\text{cr}} = 2670 \, \text{kg} \, \text{m}^{-3} \) but variable thickness, which highly elevated terrain is compensated by thick crust and low terrain or oceans by thin crust. The density of the denser mantle layer on which the mountains float is considered also to have a constant value, namely \( \rho_m = 3270 \, \text{kg} \, \text{m}^{-3} \). Thus, the density contrast between crust and mantle becomes \( \Delta \rho = \rho_m - \rho_{\text{cr}} = 600 \, \text{kg} \, \text{m}^{-3} \). A relation for the variable root (\( t \)) and anti-root (\( t' \)) thickness can be obtained from

the condition of floating equilibrium for the continents and the oceans, respectively. For flat columns one obtains, respectively

\[ t = \frac{\rho_{\text{cr}}}{\Delta \rho} \]

and

\[ t' = \left( \frac{\rho_{\text{cr}} - \rho_w}{\Delta \rho} \right) h' \]