Geoid, topography, and the Bouguer plate or shell

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Abstract. Topography plays an important role in solving many geodetic and geophysical problems. In the evaluation of a topographical effect, a planar model, a spherical model or an even more sophisticated model can be used. In most applications, the planar model is considered appropriate: recall the evaluation of gravity reductions of the free-air, Poincaré–Prey or Bouguer kind. For some applications, such as the evaluation of topographical effects in gravimetric geoid computations, it is preferable or even necessary to use at least the spherical model of topography. In modelling the topographical effect, the bulk of the effect comes from the Bouguer plate, in the case of the planar model, or from the Bouguer shell, in the case of the spherical model. The difference between the effects of the Bouguer plate and the Bouguer shell is studied, while the effect of the rest of topography, the terrain, is discussed elsewhere. It is argued that the classical Bouguer plate gravity reduction should be considered as a mathematical construction with unclear physical meaning. It is shown that if the reduction is understood to be reducing observed gravity onto the geoid through the Bouguer plate/shell then both models give practically identical answers, as associated with Poincaré’s and Prey’s work. It is shown why only the spherical model should be used in the evaluation of topographical effects in the Stokes–Helmert solution of Stokes’ boundary-value problem. The reason for this is that the Bouguer plate model does not allow for a physically acceptable condensation scheme for the topography.

Key words: Geoid – Bouguer Reduction of Gravity – Stokes–Helmert’s Problem

1 Introduction

Periodically, people discover that planar and spherical models of topography give very different results for Bouguer anomalies. Similarly, the results for the direct and indirect topographical effects in the Stokes–Helmert technique for geoid computations obtained by means of the planar and spherical models are found to be quite different. Some people claim that the planar model can safely be used for “local work” while the spherical model has to be used for global work. Others still maintain that all these questions have already been sorted out and that they do not require any more of our attention. So what is going on?

When looking into this problem (Vaníček and Novák 1999) we discovered an interesting story, which we will try to recount here. To do so, we focus only on the “infinite plate” and the “spherical shell” models, leaving out the terrain effects. The corresponding difference of the planar and spherical models of terrain presents another fascinating story which, in our opinion, requires a separate and rather more extensive paper to deal with it adequately. The main point of this “terrain story” is the discovery that, contrary to popular belief, the spherical model terrain effect has to be considered globally. This point has been already discussed by Novák et al. (1998) and by Novák and Vaníček (1999). A more formal and complete paper on the subject of spherical terrain model is under review (Novák et al. submitted). Once published, it will complement the present paper.

In the present paper, we will focus our attention first on the difference between the Bouguer plate and Bouguer shell effects on Bouguer anomalies. Then, we will tackle another, more or less independent issue, that of the difference between the two models in the gravimetric geoid determination by means of the Stokes–Helmert technique, where the use of the two models has profound consequences. In fact, the planar model, the Bouguer plate, cannot be used at all if we wish to use a physically meaningful condensation scheme.

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2 The story of Bouguer plate reduction

In order to see the pattern, let us show the gravitational potential, the gravitational attraction (negative first vertical derivative of the potential), and the vertical gradient of gravitational attraction (negative second derivative of the potential) of the topographical (Bouguer) plate and the topographical (Bouguer) shell side by side. To keep things simple, let us assume a constant density $g$ (say, the usual $2670 \text{ kg m}^{-3}$) and the same thickness, $H$, for both the infinite plate and the shell of the inner radius $R$. This is all shown in Fig. 1. The three quantities of interest are computed at two points: one on the top and one at the bottom of the plate/shell. In addition, the second derivative, which is discontinuous on the top and also at the bottom of the plate/shell, is at this point computed in both directions: from above and from below. The expressions for the plate are derived from Eqs. (3.5) and (3.7) in Heiskanen and Moritz (1967) by simply extending the finite plate to infinity. The expressions for the shell are derived directly from the equations for the potential (of a spherical shell) in Wichiencharoen [1982, Eqs. (19), (24) and (25)].

Now, examining Fig. 1, how different really are the results for the planar and spherical models? Starting from the bottom, with the vertical gradient of attraction, and neglecting the higher-order terms (of the order of $H/R$ and smaller) in the spherical model, the results are identical. The attraction of the plate at its top is only one half of that of the shell (at its top and neglecting the higher-order terms), while the attraction at the bottom of the plate is exactly opposite to that at the top. The attraction at the downside of the shell is zero, as it should be (Kellogg 1929). Note that the change in the attraction when vertically traversing the plate or shell is the same, except for higher-order terms. The situation for the potential is naturally different: as the potential of the infinite plate is infinite, we cannot make any direct comparison between the two models. We can only observe that in the spherical model, the potentials at the upside and on the downside of the shell differ only by higher-order terms.

What does it all mean? We wish to address here only the most interesting question of what this means in the context of the (incomplete, i.e. without the terrain correction) Bouguer gravity anomaly. The incomplete or simple Bouguer anomaly is computed from the following formula:

$$\Delta g = g + A + A^B - \gamma$$  \hspace{1cm} (1)

where $g$ is the observed gravity on the Earth’s surface (at altitude $H$), $A$ is the “free-air reduction” (to the geoid) due to the Earth’s masses enclosed within the inner radius of the spherical shell (including the latitude and altitude terms), $A^B$ is the “Bouguer reduction” (to the geoid) due to the mass of the Bouguer plate, and $\gamma$ is the normal gravity at the reference ellipsoid [Heiskanen and Moritz 1967, Eq. (3.19)]. In this case, the so-called Bouguer reduction is given by

$$A^B = 2 \pi G_0 H$$  \hspace{1cm} (2)

For the standard value of mean topographical density of $2670 \text{ kg m}^{-3}$, the numerical value of the Bouguer reduction is $0.1119 \times H \text{ mGal}$.

Inspecting again Fig. 1, we can see rather easily that $A^B$ is not the difference between the gravity values at the top and the bottom of the infinite plate! It is not the difference between gravity values on the upside and