The parameter distributions of the integer GPS model

P. J. G. Teunissen

Department of Mathematical Geodesy and Positioning, Delft University of Technology, Thijssenweg 11, 2629 JA Delft, The Netherlands
e-mail: p.j.g. teunissen@citg.tudelft.nl; Tel.: +31-15-278-2558; Fax: +31-15-278-3711

Received: 30 January 2001 / Accepted: 31 July 2001

Abstract. A parameter estimation theory is incomplete if no rigorous measures are available for describing the uncertainty of the parameter estimators. Since the classical theory of linear estimation does not apply to the integer GPS model, rigorous probabilistic statements cannot be made with reference to the classical results. The fact that integer parameters are involved in the estimation process forces a reappraisal of the propagation of uncertainty. It is with this purpose in mind that the joint and marginal distributional properties of both the integer and non-integer parameters of the GPS model are determined. These joint distributions can also be used to determine the distribution of functions of the parameters. As an important example, the distribution of the vector of ambiguity residuals is determined.

Key words: GPS – Ambiguity Resolution – Integer/non-integer Parameter Distributions

1 Introduction

Any GPS model of observation equations which makes use of carrier phase data and which is based on the use of two or more receivers, can be parameterized in integers and non-integers. The integer parameters refer to the unknown cycle ambiguities of the double-difference (DD) carrier phase data, while the non-integer parameters refer to the baseline components and possibly additional parameters such as atmospheric delays. When the integerness of the integer parameters is explicitly taken into account in the parameter estimation process, we speak of carrier phase ambiguity resolution. Carrier phase ambiguity resolution applies to a great variety of GPS models which are currently in use in navigation, surveying, geodesy and geophysics. An overview of these models, together with their applications, can be found in textbooks such as Leick (1995), Parkinson and Spilker (1996), Hofmann-Wellenhof et al. (1997), Strang and Borre (1997) and Teunissen and Kleusberg (1998).

As with any parameter estimation process, it is not enough to simply estimate the parameters and be done with it. We also need to have a way of inferring the uncertainty of the parameter solution. In the classical theory of linear estimation this is most often done by means of variance matrices. For the classical theory this makes sense. After all, when the model is linear and when the data are normally (Gaussian) distributed, the linear parameter estimators will be normally distributed as well. And since the peakedness of a multivariate normal distribution is completely captured by its variance matrix, it suffices to use variance matrices as a precision description of the parameters. This relatively simple approach fails to hold, however, in the case of integer GPS model. The fact that integer parameters are involved in the estimation process will result in non-Gaussian distributions even when the model is linear and the data are normally distributed. Hence, in order to obtain a rigorous description of the parameter uncertainty, we will have to bypass the use of variance matrices and go directly to the parameter distributions themselves. It is the purpose of this contribution to determine these parameter distributions.

In this contribution we use the term ‘distribution’ in the usual generic sense. For concrete cases, however, we describe the way in which the distribution of the random variate is specified. This is often in the form of a probability density function (PDF) or in the form of a probability mass function (PMF). The contribution is organized as follows. In Sect. 2 we formulate the integer GPS model and give a brief review of the principles involved in ambiguity resolution. The steps of integer estimation together with the class of admissible integer estimators are described. In Sect. 3 we determine the various parameter distributions. They hold true for any choice of integer estimator from the class of admissible ambiguity estimators. A visualization of the various
parameter distributions is also given. We first formulate our starting assumptions and then determine the joint distribution of the ‘fixed’ and ‘float’ ambiguities. This joint distribution can then be used to determine the corresponding marginal and conditional distributions. We also use it to determine the distribution of the vector of ambiguity residuals. This distribution is needed in case we want to test for the integerness of the ambiguities. Following the joint distribution of the ‘fixed’ and ‘float’ ambiguities, we determine the joint distributions of these ambiguities and the ‘float’ and ‘fixed’ baseline estimators. These distributions allow us to determine the probabilistic dependence between the baseline solution and the ambiguities. The joint distribution is then used to determine the marginal distribution of the ‘fixed’ baseline estimator. This distribution captures the complete probabilistic characteristics of the ‘fixed’ baseline. It is finally shown how this distribution can be used to construct unconditional confidence regions for the ‘fixed’ baseline.

2 The integer GPS model

Any GPS model can be cast in the following system of linear(ized) observation equations:

\[ y = Aa + Bb + e \]  

(1)

where \( y \) is the given GPS data vector of order \( m \), \( a \) and \( b \) are the unknown parameter vectors respectively of order \( n \) and \( p \), and \( e \) is the noise vector. The data vector \( y \) will usually consist of the ‘observed minus computed’ single-, dual- or triple-frequency DD phase and/or pseudorange (code) observations accumulated over all observation epochs. The entries of vector \( a \) are then the DD carrier phase ambiguities, expressed in units of cycles rather than range. They are known to be integers, \( a \in \mathbb{Z}^n \). The entries of the vector \( b \) will consist of the remaining unknown parameters, such as for instance baseline components (coordinates) and possibly atmospheric delay parameters (troposphere, ionosphere). They are known to be real-valued, \( b \in \mathbb{R}^p \). Although vector \( b \) may contain other or more real-valued unknown parameters than only those of the baseline(s), we will in this contribution, as a matter of terminology, still call its estimator the baseline estimator.

The procedure which is usually followed for solving the GPS model of Eq. (1) can be divided into three steps (Teunissen 1993). In the first step we simply discard the integer constraints \( a \in \mathbb{Z}^n \) on the ambiguities and perform a standard adjustment. As a result we obtain the (real-valued) estimates of \( a \) and \( b \), together with their variance–covariance matrix

\[
\begin{bmatrix}
\hat{a} \\
\hat{b}
\end{bmatrix}, 
\begin{bmatrix}
Q_{\hat{a}} & Q_{\hat{ab}} \\
Q_{\hat{b}a} & Q_b
\end{bmatrix}
\]  

(2)

This solution is referred to as the ‘float’ solution. In the second step the ‘float’ ambiguity estimate \( \hat{a} \) is used to compute the corresponding integer ambiguity estimate

\[
\hat{a} = S(\hat{a})
\]  

(3)

with \( S : \mathbb{R}^n \rightarrow \mathbb{Z}^n \) a mapping from the \( n \)-dimensional space of real numbers to the \( n \)-dimensional space of integers. Once the integer ambiguities are computed, they are used in the third and final step to correct the ‘float’ estimate of \( b \). As a result we obtain the ambiguity resolved baseline solution

\[
\hat{b} = \hat{b} - Q_{\hat{a}a}Q_a^{-1}(\hat{a} - \bar{a})
\]  

(4)

This solution is usually referred to as the ‘fixed’ baseline. Both Eq. (3) and Eq. (4) depend on the choice of integer estimator. Different choices of the map \( S : \mathbb{R}^n \rightarrow \mathbb{Z}^n \) will result in different integer estimators and will thus also produce differences in the probability distribution of the estimators.

There exists a whole class of integer estimators from which we can choose. In order to introduce this class, we start from the map \( S : \mathbb{R}^n \rightarrow \mathbb{Z}^n \). Due to the discrete nature of \( \mathbb{Z}^n \), the map \( S \) will not be one-to-one, but instead a many-to-one map. This implies that different real-valued ambiguity vectors will be mapped to the same integer vector. We can therefore assign a subset \( S_z \subset \mathbb{R}^n \) to each integer vector \( z \in \mathbb{Z}^n \)

\[
S_z = \{ x \in \mathbb{R}^n | z = S(x) \}, \quad z \in \mathbb{Z}^n
\]  

(5)

The subset \( S_z \) contains all real-valued ambiguity vectors that will be mapped by \( S \) to the same integer vector \( z \in \mathbb{Z}^n \). This subset is referred to as the pull-in region of \( z \) (Jonkman 1998; Teunissen 1998a). It is the region in which all ambiguity ‘float’ solutions are pulled to the same ‘fixed’ ambiguity vector \( z \). Thus \( \hat{a} = z \Leftrightarrow \hat{a} \in S_z \). By using the indicator function of the pull-in regions, the integer ambiguity estimator can be expressed as

\[
\bar{a} = \sum_{z \in \mathbb{Z}^n} z s_z(\hat{a}) \quad \text{with} \quad s_z(x) = \begin{cases} 
1 & \text{if } x \in S_z \\
0 & \text{otherwise}
\end{cases}
\]  

(6)

Since the pull-in regions define the integer estimator completely, we can define classes of integer estimators by imposing various conditions on the pull-in regions. The class of admissible integer ambiguity estimators is defined as follows.

**Definition (Admissible integer estimators)** The integer estimator \( \bar{a} = \sum_{z \in \mathbb{Z}^n} z s_z(\hat{a}) \) is said to be admissible when its pull-in regions \( S_z = \{ x \in \mathbb{R}^n | z = S(x) \}, z \in \mathbb{Z}^n \), satisfy

\[
(i) \quad \cup_{z \in \mathbb{Z}^n} S_z = \mathbb{R}^n \\
(ii) \quad \text{Int } S_{z_1} \cap \text{Int } S_{z_2} = \emptyset, \quad \forall z_1, z_2 \in \mathbb{Z}^n, z_1 \neq z_2 \\
(iii) \quad S_z = z + S_0, \quad \forall z \in \mathbb{Z}^n
\]  

(7)

where ‘Int’ denotes the interior of the subset. For the motivation of the above definition we refer to Teunissen (1999). Examples of integer estimators that belong to the above class are integer rounding, integer bootstrapping and integer least-squares (LS). For the material that follows, it is useful to note that the indicator functions of admissible pull-in regions fulfill the two basic conditions of PMFs and PDFs, namely of being non-negative and having an area of 1. For fixed \( x \in \mathbb{R}^n \), \( s_z(x) \) fulfills the PMF conditions