Exposita Notes

Two new proofs of Afriat’s theorem

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Summary. We provide two new, simple proofs of Afriat’s celebrated theorem stating that a finite set of price-quantity observations is consistent with utility maximization if, and only if, the observations satisfy a variation of the Strong Axiom of Revealed Preference known as the Generalized Axiom of Revealed Preference

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1 Introduction

The neoclassical theory of demand supposes that a consumer, facing a price vector \( p \in \mathbb{R}^\ell_+ \) and with income \( I > 0 \), chooses his demand bundle \( x \in \mathbb{R}^\ell_+ \) to maximize some utility function \( u : \mathbb{R}^\ell_+ \to \mathbb{R} \) over his budget set \( B(p, I) := \{ x \in \mathbb{R}^\ell_+ : p \cdot x \leq I \} \). We assume we have been presented with a finite data set \( D := \{(p_i, x_i) : i \in N\} \), where \( N := \{1, 2, \ldots, n\} \), of price vectors \( p_i \in \mathbb{R}^\ell_+ \) and corresponding demand vectors \( x_i \in \mathbb{R}^\ell_+ \). The basic question raised by Afriat is whether this data set is consistent with the maximization of a locally non-satiated utility function \( u \) in the sense that for each \( i \in N \), \( x_i \) maximizes \( u \) over \( B(p_i, p_i \cdot x_i) \). A locally non-satiated utility function is one for which every neighborhood of a commodity bundle contains another bundle with a higher utility. With such a utility function the consumer will have spent all his income, so that we can use \( p_i \cdot x_i \) as the income for situation \( i \).

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If the set of price and quantity observations is derived from utility maximization it will surely satisfy the variation of the Strong Axiom of Revealed Preference, known as the Generalized Axiom of Revealed Preference, which states that, for any list \((x_1, p_1), \ldots, (x_n, p_n)\) with the property that
\[ p_j \cdot x_{j+1} \leq p_j \cdot x_j, \text{ for all } j \leq n - 1, \]
we must have \(p_n \cdot x_1 \geq p_n \cdot x_n.\)

The argument for the Generalized Axiom is straightforward. If \(p_j \cdot x_{j+1} \leq p_j \cdot x_j\) then \(x_{j+1}\) could have been purchased at prices \(p_j\). Since \(x_{j+1}\) was not purchased it cannot be strictly preferred to \(x_j\) so that \(x_j \succeq x_{j+1}\). The entire sequence of inequalities therefore implies that \(x_1 \succeq x_n\). If, on the other hand, \(p_n \cdot x_1 < p_n \cdot x_n\) and the utility function is locally non-satiated, we could find a commodity bundle \(\xi\) close to \(x_1\) with \(p_n \cdot \xi < p_n \cdot x_n\) and \(\xi \succ x_n\), violating the assumption that \(x_n\) maximizes utility at prices \(p_n\) and income \(p_n \cdot x_n\).

The Generalized Axiom may be stated in a slightly different fashion which is more appropriate for our needs. If the inequalities
\[ p_j \cdot x_{j+1} \leq p_j \cdot x_j, \text{ hold for all } j \leq n - 1 \text{ and if} \]
\[ p_n \cdot x_1 \leq p_n \cdot x_n \text{ as well}, \]
then we must have \(p_n \cdot x_1 = p_n \cdot x_n\). But in this form there is no distinction between the last observation and any of the other observations, so that
\[ p_j \cdot x_{j+1} = p_j \cdot x_j \]
holds for all \(j\). This is the variation of the Strong Axiom which we shall adopt, not only for the full set of \(n\) observations but for any ordered subset as well.

**Definition 1** We say that the observations satisfy the Generalized Axiom of Revealed Preference (GARP) if for every ordered subset \(\{i, j, k, \ldots, r\} \subseteq \mathbb{N}\) with
\[ p_i \cdot x_j \leq p_i \cdot x_i \]
\[ p_j \cdot x_k \leq p_j \cdot x_j \]
\[ \vdots \]
\[ p_r \cdot x_i \leq p_r \cdot x_r \]
it must be true that each inequality is, in fact, an equality.

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1 There is a great variety of terminology associated with the concept of revealed preference. The original definition offered by Samuelson [4], now known as the Weak Axiom of Revealed Preference (WARP), was thought by the author to be sufficient to recover a utility function generating the data. Houthakker’s definition of the Strong Axiom (SARP) [3] provided the additional conditions necessary for recovery. But Houthakker’s statement of the Strong Axiom is motivated by a single valued demand function rather than a finite list of observations and is, as a consequence, somewhat awkward. Afriat [1] used the terminology Cyclical Consistency (CC) for the simpler concept of the current paper. Cyclical Consistency is identical with the Generalized Axiom of Revealed Preference (GARP) introduced by Varian [5]. This does not exhaust the list of variations in terminology.

We have chosen to use the term GARP rather than Cyclical Consistency. Our purpose is to use a definition in which the phrase “Revealed Preference” actually appears rather than the earlier, equivalent terminology used by Afriat.