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Unanimous subjective probabilities

Abstract This note shows that if the space of events is sufficiently rich and the subjective probability function of each individual is non-atomic, then there is a $\sigma$-algebra of events over which everyone will have the same probability function, and moreover, the range of this common probability is the entire unit interval.

Keywords Agreement · Subjective probability · Objective probability

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1 Introduction

An important assumption in social choice theory is the existence of social lotteries, that is, lotteries whose outcomes are social policies.¹ Such lotteries can increase the fairness of the social allocation mechanism or solve disputes in a cheap, efficient manner. For a social lottery to be acceptable, it must be considered fair by all

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¹ See Harsanyi (1955), and more recently, Epstein and Segal (1992), Broome (1984), Kamm (1993–1996), or Karni and Safra (2002).
individuals in society. In particular, if society finds it optimal to randomize over the $k$ pure social policies $s_1, \ldots, s_k$ by using the probability vector $p = (p_1, \ldots, p_k)$, then everyone in society must agree that the mechanism is indeed using these probabilities$^2$.

But do such mechanisms exist? Diamond (1967) thought that when probabilities are subjective, the answer is no. Even in the model of Anscombe and Aumann (1963), where each decision maker is assumed to face subjective “horse race” lotteries and objective “roulette wheels,” it does not follow that all decision makers agree on what is objective. An Italian-speaking person, facing a die whose sides are marked Uno, Tre, Cinque, Sette, Otto, Dieci will consider the event “the die will show an odd number” to be objective, while a non Italian-speaking person will consider it subjective (or even ambiguous). Nothing in the assumptions and structure of the Anscombe–Aumann model implies agreement on what constitutes a roulette lottery. The issue is even more critical in Savage (1954) framework, where all events are assumed to be subjective.

Recently, Ghirardato et al. (2003) showed that even if probabilities do not exist (that is, beliefs are ambiguous), it is still possible, under some assumptions, to obtain mixture-like operators over random variables. But these procedures are subjective, and cannot be jointly used. Machina (2004), on the other hand, assumes that preferences are smooth and proves that for each $r \in [0, 1]$ there is a sequence of events $E_n$ such that for each $i, \mu_i(E_n) \to r$. Unfortunately, as noted by Machina, the limits of these sequences of events don’t necessarily exist.$^3$ Moreover, from the social point of view it may be important for everyone to agree that an event has probability exactly $1/n$, not just approximately $1/n$.

In this note we show that if the space of events is sufficiently rich and the subjective probability function of each individual is non-atomic, then there is a $\sigma$-algebra of events over which everyone will have the same probability function, and moreover, the range of these probabilities is the entire interval $[0, 1]$. In other words, even in a fully subjective world (for example, Savage’s), there is a rich set of events that can be used for joint randomization. We prove existence, but we do not yet know how to construct a specific such $\sigma$-algebras. This does not void the contribution of this note. Randomization in social choice theory plays an important theoretical role, but it does not follow that policy makers do randomize. Our aim is to close a theoretical gap that exists in the literature—if commonly accepted devices do not exist, then models using randomization to enhance fairness would become void. Theorem 1 shows that there are enough events over which decision makers agree.

2 A Theorem

**Theorem 1** Let $\mu_1, \ldots, \mu_n$ be nonatomic, countably additive probability measures on a measurable space $(S, \Sigma)$. Then there is a sub-$\sigma$-algebra $\hat{\Sigma}$ of $\Sigma$ on

$^2$ Unlike the cake division problem, where participants want to ensure receiving at least their fair share, in social choice individuals often wish also not to receive more than their fair share.

$^3$ For example, let the event $A_n$ be “the $n$-th digit to the right of the decimal point of the temperature tomorrow will be odd.” Then as $n \to \infty$, all individual beliefs regarding these events will converge to $1/2$. But there is no sense of limit for which lim $A_n$ exists as an event of probability $1/2$. 