Equilibrium tracing in strategic-form games

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Abstract We analyze the relationships of the van den Elzen–Talman algorithm, the Lemke–Howson algorithm and the global Newton method for equilibrium computation by Govindan and Wilson. For two-player games, all three can be implemented as complementary pivoting algorithms. The algorithms by Lemke and Howson and by van den Elzen and Talman start at a pair of strategies: the first method at a pure strategy and its best reply, the latter anywhere in the strategy space. However, we show that even with the same starting point they may find different equilibria. Our second result is that the van den Elzen–Talman algorithm is a special case of the global Newton method, which was known only for the Lemke–Howson algorithm. More generally, the global Newton method implements the linear tracing procedure for any number of players. All three algorithms find generically only equilibria of positive index. Even though the van den Elzen–Talman algorithm is extremely flexible in the choice of starting point, we show that there are generic coordination games where the completely mixed equilibrium, which has positive index, is generically not found by the algorithm.

Keywords Bimatrix game · Equilibrium computation · Homotopy methods · Index

JEL Classification C72

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1 Introduction

In this article, we investigate several algorithms for the computation of Nash equilibria in strategic-form games. The algorithms by Lemke and Howson (1964) and van den Elzen and Talman (1991) for bimatrix games are complementary pivoting methods; both have been studied extensively. The difference between the two methods is that while the Lemke–Howson method only allows for a restricted (finite) set of paths, the van den Elzen–Talman algorithm can start at any mixed strategy pair, called prior, and hence allows to generate infinitely many paths. This implies that the van den Elzen–Talman algorithm is more flexible than the Lemke–Howson method. An even more versatile algorithm is the global Newton method by Govindan and Wilson (2003a), which works for finite strategic-form games. All three algorithms can be interpreted as homotopy methods, see Herings and Peeters (2009).

We investigate the relations between these three algorithms. We show that the Lemke–Howson and van den Elzen–Talman algorithms differ substantially: The van den Elzen–Talman algorithm, when started from a pure strategy and its best response as a prior, in general finds a different equilibrium than the corresponding Lemke–Howson method. This is not surprising since both algorithms can be understood as special cases of the global Newton method, but in very different ways. For the van den Elzen–Talman algorithm, the description as a special case of the global Newton method seems to be a new result, which we generalize to the statement that for $N$-player strategic-form games, the global Newton method implements the linear tracing procedure introduced by Harsanyi (1975).

As a special case of the global Newton method, the van den Elzen–Talman algorithm can generically find only equilibria of index $+1$. This leads us to the issue of traceability of equilibria. Following Hofbauer (2003), we call an equilibrium in a bimatrix game traceable if it is found by the van den Elzen–Talman algorithm from an open set of priors. As explained above, the van den Elzen–Talman algorithm allows for much greater flexibility than the Lemke–Howson method. Hence one might hope that, unlike the Lemke–Howson algorithm, it is powerful enough to find all equilibria of index $+1$. This raises the until now open question if, generically, all equilibria of index $+1$ are traceable. We answer this question negatively by analyzing traceability in coordination games.

If a nondegenerate $3 \times 3$ coordination game has a completely mixed equilibrium, this equilibrium has index $+1$. In addition, the game has three pure strategy equilibria, also of index $+1$, and three equilibria of support size two, which have index $-1$. Hofbauer (2003) noted that in a symmetric $3 \times 3$ coordination game, the completely mixed equilibrium (if it exists) is not traceable. We show that, in general, this is only correct as long as we restrict the starting points of the van den Elzen–Talman paths to symmetric strategy profiles. More precisely, we will see that the traceability of the completely mixed equilibrium in a $3 \times 3$ coordination game depends on the specific geometry of the best reply regions. We prove that for certain generic coordination games the completely mixed equilibrium is traceable. However, we also show that there is a generic set of coordination games whose completely mixed equilibrium is not traceable. Hence there is an open set in the space of $3 \times 3$ bimatrix games that all have an untraceable equilibrium of index $+1$. This implies that the flexibility of the