Abstract

We study the existence of strong solutions to the three-dimensional Navier-Stokes initial-boundary value problem in the domain, $\Omega$, exterior to a rigid body that rotates with constant angular velocity, $\omega$. We show that when the initial data, $u_0$, are prescribed in an appropriate functional class, a strong solution exists at least in some finite time interval. Moreover, the solution exists for all times, provided $u_0$, in suitable norm, and the magnitude of $\omega$ do not exceed a certain constant depending only on the kinematic viscosity and on the regularity of $\Omega$. In this latter case, we also show that the velocity field converges to the velocity field of the corresponding steady-state solution.

1. Introduction

This paper is devoted to the unsteady flow of a viscous incompressible fluid, $\mathcal{L}$, past a rotating rigid body, $B$. We assume that the motion of $\mathcal{L}$ is governed by the Navier-Stokes equations. We also suppose that the angular velocity of $B$ is known in an inertial reference frame and that, with respect to that frame, it is independent of time.

As is well known, this type of problem arises in many significant engineering studies that involve rigid bodies moving in a viscous liquid, on both small and large scales; see, e.g., [6] and the literature cited therein.

In order to make the region occupied by $\mathcal{L}$ time-independent, it is convenient to refer the relevant equations to a frame $\mathcal{R}$ attached to $B$. Thus, denoting by $u = (u_1, u_2, u_3)$ and $p$ the velocity field and the associated pressure field of $\mathcal{L}$ in $\mathcal{R}$, and by $\omega$ the angular velocity of $B$ in $\mathcal{R}$, the generic unsteady motion of $\mathcal{L}$ is governed by the following initial-boundary value problem (see, e.g., [10, 6])
\[ \begin{align*}
\partial_t u + u \cdot \nabla u &= \nu \Delta u - \nabla p + \omega \times x \cdot \nabla u - \omega \times u \quad \text{in } \Omega \times [0, \infty[ , \\
\nabla \cdot u &= 0 \\
\lim_{|x| \to \infty} u(x, t) &= 0, \quad t \in [0, \infty[ , \\
u \Delta u_S - \nabla p^S &= u^S \cdot \nabla u^S + \omega \times u^S - \omega \times x \cdot \nabla u^S \quad \text{in } \Omega , \\
u \Delta u_S - \nabla p^S &= u^S \cdot \nabla u^S + \omega \times u^S - \omega \times x \cdot \nabla u^S \quad \text{in } \Omega , \\
\nabla \cdot u^S &= 0, \\
\lim_{|x| \to \infty} u^S(x) &= 0 .
\end{align*} \]

In these equations, the (three-dimensional) exterior domain \( \Omega \) denotes the (time-independent) region of flow of \( L \) and \( \Sigma \) represents the boundary of \( \Omega \). It is supposed throughout that \( \Sigma \) is of class \( C^2 \).

Along with (1), we shall also consider the corresponding steady problem for the fields \( u^S, p^S \):

\[ \begin{align*}
u \Delta u^S - \nabla p^S &= u^S \cdot \nabla u^S + \omega \times u^S - \omega \times x \cdot \nabla u^S \quad \text{in } \Omega , \\
u \Delta u^S - \nabla p^S &= u^S \cdot \nabla u^S + \omega \times u^S - \omega \times x \cdot \nabla u^S \quad \text{in } \Omega , \\
\nabla \cdot u^S &= 0, \\
\lim_{|x| \to \infty} u^S(x) &= 0 .
\end{align*} \]

Problems (1) and (2) have received the attention of several authors. Before listing the main contributions, we would like to emphasize an essential point: what makes these problems difficult is the presence of the term \( \omega \times x \cdot \nabla u \), whose coefficient becomes unbounded at large spatial distances. This implies that \( -\omega \times u + \omega \times x \cdot \nabla u \) is not a perturbation to the linear (Stokes) operator, even for “small” \( |\omega| \).

Concerning the initial-boundary value problem (1), BORCHERS showed the existence of global weak solutions à la Leray-Hopf [2]. The asymptotic behavior in time of such solutions was successively explored by CHEN & MIYAKAWA [3], in the case when \( \Omega \) is the whole space (Cauchy problem). The first result of existence of more regular solutions is due to HISHIDA [10]. Specifically, employing a non-trivial generalization of the semigroup method of FUJITA & KATO [4], he showed the existence and uniqueness of a “mild” solution, provided the initial data \( u_0 \) belong to the domain of a suitable fractional power of the Stokes operator. Existence is only established in a finite interval of time, \([0, T]\), \( T > 0 \). The velocity field \( u \) of these solutions is continuous in \([0, T]\) with values in the Sobolev space \( H^2(\Omega) \). However, as the author himself remarks (see [10] p.314) the differentiability of \( u \) with respect to time is not known, and he conjectures that, in order to show that \( \partial_t u \) exists as an element of the Lebesgue space \( L^2(\Omega) \), it is necessary for \( \omega \times x \cdot \nabla u_0 \) to belong to suitable Sobolev spaces. One consequence of this fact is that very little can be said about the regularity of the pressure. Recently, an initial-boundary value problem more general than (1), where \( B \) is allowed to move by a generic (and unknown) rigid motion, has been studied by GALDI & SILVESTRE [8]. There, the authors establish the existence of strong solutions but, as in Hishida’s work, only in