Incompressible Viscous Flows in Borderline Besov Spaces

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Abstract

We establish two new estimates for a transport-diffusion equation. As an application we treat the problem of global persistence of the Besov regularity \( B_{\frac{2}{p}+1}^{\frac{2}{p}+1} \), with \( p \in [2, +\infty] \), for the two-dimensional Navier–Stokes equations with uniform bounds on the viscosity. We provide also an inviscid global result.

1. Introduction

In this paper we are concerned with the incompressible Navier–Stokes equations governing a viscous fluid evolving in the whole space \( \mathbb{R}^d \),

\[
\begin{align*}
(\text{NS}_\mu) \quad \begin{cases}
\partial_t v_\mu + v_\mu \cdot \nabla v_\mu - \mu \Delta v_\mu = -\nabla \pi_\mu \\
div v_\mu = 0 \\
v_\mu(0, x) = v^0(x).
\end{cases}
\end{align*}
\]

Here, the vector field \( v_\mu(t, x) = (v_\mu^1, \ldots, v_\mu^d)(t, x) \) stands for the velocity of the fluid, the scalar \( \pi_\mu \) denotes the pressure and the parameter \( \mu > 0 \) is the kinematic viscosity. We will also consider the Euler equations which are the inviscid case of the system \( (\text{NS}_\mu) \),

\[
(\text{E}) \quad \begin{cases}
\partial_t v + v \cdot \nabla v = -\nabla \pi \\
div v = 0 \\
v(0, x) = v^0(x).
\end{cases}
\]

The Cauchy problem for Navier–Stokes system has been intensively investigated since the pioneering work of LERAY [12] who proved the existence of global weak solutions in the energy space. Nevertheless, the uniqueness of such solutions is only known in space dimension two and is still a widely open problem for higher dimension \( (d \geq 3) \). For strong solutions, FUJITA and KATO [7] proved a local
well-posedness result when the initial data are lying in the homogeneous Sobolev space $H^{d-1}$, which is invariant under the scaling of the equations. It is worth pointing out that the same result holds true when the initial data belong to the inhomogeneous Sobolev space $H^s$, with $s \geq \frac{d}{2} - 1$. However the problem of whether these solutions blow up in finite time or not is still unsolved and is considered as one of the most relevant problem of the nonlinear PDE’s. We emphasize that the global existence is only known in some restrictive cases as for example in space dimension two or when the initial data are small in some function spaces which are invariant under the scaling.

For a Euler system the theory is widely developed, and we restrict our attention to some significant results. In [11] Kato and Ponce proved a local well-posedness result in $H^s$ with $s > \frac{d}{2} + 1$. In space dimension two such solutions are global since the vorticity does not concentrate in finite time, see for instance [2]. It seems very hard to obtain a local well-posedness result when initial data belong to the critical space $H^{1+d/2}$ because we lack the embedding in Lipschitz functions. Nevertheless local results are given when we work with Besov spaces $B^{1+d/p}_{p,1}$, with $p \in [1, \infty]$, see [3]. Recently, Vishik has proved in [15] that these solutions are global in dimension two. His proof relies on a subtle logarithmic estimate based on the explicit formula of the vorticity in dimension two. We mention that we have extended in [10] the global existence in the limiting case $p = +\infty$.

The inviscid limit was performed by several authors and is well understood. For example, it is proven in [13] that under the assumption $\nu^0 \in H^s$ with $s > \frac{d}{2} + 2$, the solutions $(\nu^\mu)_{\mu > 0}$ converge in $L^2$ norm as $\mu \to 0$ to the unique solution $\nu$ of $(E)$ and the convergence rate is of order $\mu t$. We point out that in dimension two these results are global in time.

This paper is the sequel to [9] in which we continue to study essentially two problems. The first one is the uniform persistence with respect to vanishing viscosity of viscous solutions in critical Besov spaces $B^{2/p+1}_{p,1}({\mathbb{R}}^2)$, whereas the second one deals with the inviscid limit and the rate convergence. In [9] we have given an affirmative answer when $p \in [1, 2]$ and the crucial fact of the proof is a new regularization effect of the vorticity equation:

$$\partial_t \omega^\mu + \nu^\mu \cdot \nabla \omega^\mu - \mu \Delta \omega^\mu = 0 \quad \text{with} \quad \omega^\mu = \partial_1 v_2^\mu - \partial_2 v_1^\mu.$$  

We recall the following identity:

$$v^\mu = \Delta^{-1} \nabla^\perp \omega^\mu \quad \text{with} \quad \nabla^\perp = (-\partial_2, \partial_1). \quad (1)$$

We have established in [9] the following linear estimate on the Lipschitz norm of the velocity

$$\mu \left\| \int_0^t \omega^\mu(\tau) \, d\tau \right\|_{B^{1+d}_{\infty,\infty}} \leq C \|\omega^0\|_{L^\infty} \left(1 + \int_0^t \|\nu v^\mu(\tau)\|_{L^\infty} \, d\tau \right),$$

which is essential to bound uniformly on $\mu$ the quantity $\|\nabla v^\mu\|_{L^1 L^\infty}$ (for the definition of Besov spaces we refer the reader to the next section). This method fails for $p > 2$, and we are led to use another approach based on a new logarithmic estimate. Our main result reads as follows.