Mixed Boundary Value Problems for the Stationary Navier–Stokes System in Polyhedral Domains

V. Maz’ya & J. Rossmann

Communicated by V. Sverak

Abstract

The authors consider boundary value problems for the Navier–Stokes system in a polyhedral domain, where different boundary conditions (in particular, Dirichlet, Neumann, slip conditions) are arbitrarily combined on the faces of the polyhedron. They prove existence and regularity theorems for weak solutions in weighted (and nonweighted) $L^p$ Sobolev and Hölder spaces with sharp integrability and smoothness parameters.

1. Introduction

Steady-state flows of incompressible viscous Newtonian fluids are modeled by the Navier–Stokes equations

$$\begin{align*}
- \nu \Delta u + (u \cdot \nabla) u + \nabla p &= f, \\
\nabla \cdot u &= g
\end{align*}$$

(1.1)

for the velocity $u$ and the pressure $p$. For this system, one can consider different boundary conditions. For example on solid walls, we have the Dirichlet condition $u = 0$. On other parts of the boundary (an artificial boundary such as the exit of a channel, or a free surface) a no-friction condition (Neumann condition) $2\nu \varepsilon(u) n - pn = 0$ may be useful. Here $\varepsilon(u)$ denotes the matrix with the components $\frac{1}{2}(\partial_x u_j + \partial_x u_i)$, and $n$ is the outward normal. Note that the Neumann problem naturally appears in the theory of hydrodynamic potentials (see [12]). It is also of interest to consider boundary conditions containing components of the velocity and of the friction. Frequently used combinations are the normal component of the velocity and the tangential component of the friction (slip condition for uncovered fluid surfaces) or the tangential component of the velocity and the normal component of the friction (condition for in/out-stream surfaces).

In the present paper, we consider mixed boundary value problems for the system (1.1) in a three-dimensional domain $\mathcal{G}$ of polyhedral type, where components of
the velocity and/or the friction are given on the boundary. To be more precise, we have one of the following boundary conditions on each face $\Gamma_j$:

(i) $u = h$,
(ii) $u_\tau = h, -p + 2\nu \varepsilon_{n,n}(u) = \phi$,
(iii) $u_n = h, 2\nu \varepsilon_{n,n}(u) = \phi$,
(iv) $-pn + 2\nu \varepsilon_n(u) = \phi$,

where $u_n = u \cdot n$ denotes the normal and $u_\tau = u - u_n n$ the tangential component of $u$, $\varepsilon_n(u)$ is the vector $\varepsilon(u) n$, $\varepsilon_{n,n}(u)$ is the normal component and $\varepsilon_{n,\tau}(u)$ the tangential component of $\varepsilon_n(u)$.

Weak solutions, that is, variational solutions $(u, p) \in W^{1,2}(\mathcal{G})^3 \times L^2(\mathcal{G})$, always exist if the data are sufficiently small. In the case when the boundary conditions (ii) and (iv) disappear, such solutions exist for arbitrary $f$ (see the books by Ladyzhenskaya [12], Temam [25], Girault & Raviart [5]). Our goal is to prove regularity assertions for weak solutions. As is well known, the local regularity result

$$(u, p) \in W^{l,s} \times W^{l-1,s}$$

is valid outside an arbitrarily small neighborhood of the edges and vertices if the data are sufficiently smooth. Here $W^{l,s}$ denotes the Sobolev space of functions which belong to $L^s$ together with all derivatives up to order $l$. The same result holds for the Hölder space $C^{l,\sigma}$. Since solutions of elliptic boundary value problems in general have singularities near singular boundary points, the result cannot be globally true in $\mathcal{G}$ without any restrictions on $l$ and $s$. Here we give a few particular regularity results which are consequences of more general theorems proved in the present paper. Suppose that the data belong to corresponding Sobolev or Hölder spaces and satisfy certain compatibility conditions on the edges. Then the following smoothness of the weak solution is guaranteed and is the best possible.

- If $(u, p)$ is a solution of the Dirichlet problem in an arbitrary polyhedron or a solution of the Neumann problem in an arbitrary Lipschitz graph polyhedron, then

$$(u, p) \in W^{1,3+\varepsilon}(\mathcal{G})^3 \times L^{3+\varepsilon}(\mathcal{G}),$$
$$(u, p) \in W^{2,4/3+\varepsilon}(\mathcal{G})^3 \times W^{1,4/3+\varepsilon}(\mathcal{G}),$$
$$u \in C^{0,\varepsilon}(\mathcal{G})^3.$$

Here $\varepsilon$ is a positive number depending on the domain $\mathcal{G}$.

- If $(u, p)$ is a solution of the Dirichlet problem in a convex polyhedron, then

$$(u, p) \in W^{1,s}(\mathcal{G})^3 \times L^{s}(\mathcal{G}) \quad \text{for all } s, 1 < s < \infty,$$
$$(u, p) \in W^{2,2+\varepsilon}(\mathcal{G})^3 \times W^{1,2+\varepsilon}(\mathcal{G}),$$
$$(u, p) \in C^{1,\varepsilon}(\mathcal{G})^3 \times C^{0,\varepsilon}(\mathcal{G}).$$

- If $(u, p)$ is a solution of the mixed problem in an arbitrary polyhedron with the Dirichlet and Neumann boundary conditions prescribed arbitrarily on different faces, then

$$(u, p) \in W^{2,8/7+\varepsilon}(\mathcal{G})^3 \times W^{1,8/7+\varepsilon}(\mathcal{G}).$$