Carleman Estimate for Elliptic Operators with Coefficients with Jumps at an Interface in Arbitrary Dimension and Application to the Null Controllability of Linear Parabolic Equations

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Abstract

In a bounded domain of $\mathbb{R}^{n+1}$, $n \geq 2$, we consider a second-order elliptic operator, $A = -\partial^2_{x_0} - \nabla \cdot (c(x)\nabla_x)$, where the (scalar) coefficient $c(x)$ is piecewise smooth yet discontinuous across a smooth interface $S$. We prove a local Carleman estimate for $A$ in the neighborhood of any point of the interface. The “observation” region can be chosen independently of the sign of the jump of the coefficient $c$ at the considered point. The derivation of this estimate relies on the separation of the problem into three microlocal regions and the Calderón projector technique. Following the method of Lebeau and Robbiano (Comm Partial Differ Equ 20:335–356, 1995) we then prove the null controllability for the linear parabolic initial problem with Dirichlet boundary conditions associated with the operator $\partial_t - \nabla \cdot (c(x)\nabla_x)$.

Contents

1. Introduction and notation .............................. 953
2. Local Carleman estimate at the interface ...................... 957
   2.1. Preliminaries .................................. 960
   2.2. Signs of the imaginary part of the two roots of $p_{\delta}^{(d)}$ ........................... 962
   2.3. Estimate in the region $\mathcal{E}^{d,+}$ ........................... 967
   2.4. Estimate in the region $\mathcal{E}^{d,-}$ ........................... 975
   2.5. Estimate around the region $\mathcal{Z}^{d}$ ......................... 979
   2.6. Proof of Theorem 2.1 .............................. 984
3. Application to null controllability .......................... 984
   3.1. Proof of the interpolation inequality .......................... 985

1. Introduction and notation

The question of the null controllability of linear parabolic partial differential equations with smooth coefficients was solved in the 1990s [7,14]. In the case of
discontinuous coefficients in the principal part of the parabolic operator, the con-
trollability issue and its dual counterpart, observability, are not fully solved yet. A result of controllability for a semi-linear heat equation with a coefficient that is discontinuous at an interface was proven in [5] by means of a global Carleman observability estimate. Roughly speaking, as in the case of hyperbolic systems (see for example [13, page 356]), the authors of [5] proved their controllability result in the case where the control is supported in the region where the diffusion coefficient is the “lowest”. In both cases, however, the approximate controllability, and its dual counterpart, uniqueness, are true without any restriction on the monotonicity of the coefficients. It is then natural to question whether or not an observability estimate holds in the case of non-smooth coefficients and arbitrary observation location.

Recently, in the one-dimensional case, the controllability result for parabolic equations was proven for general piecewise $C^1$ coefficients in [2], and for coefficients with bounded variations (BV) in [12], which improved the result of [6]. The proof relies on global Carleman estimates, which moreover allow to treat semi-linear equations. Simultaneously, a controllability result for parabolic equations with general bounded coefficients in one dimension was proven in [1]. The method used there to achieve null controllability is that of [14], which limits the field of applications to linear equations.

In the $n$-dimensional case, $n \geq 2$, a positive answer to the controllability question was given for a class of discontinuous coefficients, with separated variables, that are smooth with respect to all but one variables, which includes the case of stratified media [3]. The proof relies both on the Carleman estimates of [2,12] in the one-dimensional case and the method of [14].

In the present article, in the case $n \geq 2$, we achieve null controllability for a linear parabolic equation in the case of a coefficient that exhibits jumps of arbitrary signs at an interface. Let $\Omega$ be a smooth bounded connected domain in $\mathbb{R}^n$. We consider the operator $L := \nabla_x \cdot (c(x)\nabla_x)$, with possibly additional lower-order terms, and where $c(x)$ satisfies
\[ 0 < c_{\min} \leq c(x) \leq c_{\max} < \infty, \]
to ensure uniform ellipticity for $L$. The coefficient $c$ is assumed smooth apart from across an interface $S$, where it may jump. The interface $S$ is the boundary of a smooth open subset $\Omega_1 \Subset \Omega$, that is, $\Omega_1$ lies on one side of $S$. Let $T > 0$ and set $Q_T = (0, T) \times \Omega$. We set $\Omega_2 = \Omega \setminus \overline{\Omega_1}$. We prove the following null controllability result.

**Theorem 1.1.** For an arbitrary time $T > 0$ and an arbitrary non-empty open subset $\omega \subset \Omega$ and an initial condition $q_0 \in L^2(\Omega)$, there exists $u \in L^2((0, T) \times \Omega)$ such that the solution $q$ of
\[
\begin{align*}
\partial_t q - Lq &= 1_{\omega}u \quad \text{in } Q_T, \\
q(t, x) &= 0 \quad \text{on } (0, T) \times \partial \Omega, \\
q(0, x) &= q_0(x) \quad \text{in } \Omega,
\end{align*}
\]
satisfies $q(T) = 0$ almost everywhere in $\Omega$. 