Regularity of Optimal Maps on the Sphere: the Quadratic Cost and the Reflector Antenna

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Abstract

Building on the results of MA et al. (in Arch. Rational Mech. Anal. 177 (2), 151–183 (2005)), and of the author Loeper (in Acta Math., to appear), we study two problems of optimal transportation on the sphere: the first corresponds to the cost function $d^2(x, y)$, where $d(\cdot, \cdot)$ is the Riemannian distance of the round sphere; the second corresponds to the cost function $-\log |x - y|$, known as the reflector antenna problem. We show that in both cases, the cost-sectional curvature is uniformly positive, and establish the geometrical properties so that the results of Loeper (in Acta Math., to appear) and MA et al. (in Arch. Rational Mech. Anal. 177 (2), 151–183 (2005)) can apply: global smooth solutions exist for arbitrary smooth positive data and optimal maps are Hölder continuous under weak assumptions on the data.

1. Introduction

1.1. Monge–Kantorovitch problem on a Riemannian manifold

Let $M$ be a topological space, let $c : M \times M \to \mathbb{R} \cup \{+\infty\}$ be a cost function and $\mu_0, \mu_1$ be probability measures on $M$. In the optimal transportation problem, one looks for a map $T : M \to M$ that minimizes the functional

$$I(T) = \int_M c(x, T(x))d\mu_0(x),$$

under the constraint that $T$ pushes forward $\mu_0$ onto $\mu_1$, (hereafter $T_#\mu_0 = \mu_1$), that is

$$\forall B \subset M \text{ Borel}, \quad \mu_1(B) = \mu_0(T^{-1}(B)).$$
This problem was first studied by Monge [19], with $M = \mathbb{R}^n$, for the cost $c = |x - y|$ (the Euclidean distance). Beyond Monge’s original problem, Brenier studied the case of the quadratic cost $c = |x - y|^2$, and pointed out its close connection with important nonlinear PDEs (Monge–Ampere, Euler etc...). For the quadratic cost, when $\mu_0$ is absolutely continuous with respect to the Lebesgue measure, he proved the existence and uniqueness of an optimal map $T$. This map has a convex potential (that is $T = \nabla \phi$ with $\phi$ convex) and is shown to be the only map with convex potential that pushes $\mu_0$ forward onto $\mu_1$. Since Brenier’s result, the theory of optimal transportation has been extended to general cost functions. The existence of optimal maps is granted under very generic conditions on the cost function, and the way to obtain it is achieved through a general procedure, known as the Kantorovitch duality: Optimal maps are obtained by solving the dual Monge–Kantorovitch problem, whose unknowns are potential functions. For $\phi$, a lower semi-continuous function on $M$, we define its $c$-transform as

$$\phi^c(x) = \sup_{y \in M} \{-c(x, y) - \phi(x)\}.$$ 

A potential $\phi$ is $c$-convex if it is the $c$-transform of some other $\psi : M \to \mathbb{R}$. In that case, the equality $\phi = \phi^{cc}$ holds. (Notice that the quadratic cost is equivalent to the cost $-x \cdot y$, for which the $c$-transform is nothing but the Legendre–Fenchel transform; hence, $[-x \cdot y]$-convex functions are convex functions.) Under suitable assumptions, and following, for example [6], the minimizers in (1) are related to $c$-convex potentials as follows: for an optimal $T_{\text{opt}}$ in (1), there exists a $c$-convex potential $\phi$ such that

$$\text{for a.e. } x \in M, T_{\text{opt}}(x) = G_\phi(x) := \{y \in M, \phi(x) + \phi^c(y) = -c(x, y)\}. \quad (2)$$

Conversely, if $T : M \to M$ can be expressed under the form (2) for some $c$-convex $\phi$, for $\mu_0$ a probability measure on $M$ and $\mu_1$ its push-forward by $T$, then $T$ is the optimal map between $\mu_0$ and $\mu_1$. Of course, it is not clear a-priori that (2) defines a map, as $G_\phi(x)$ is a set. However, under a suitable assumption on the cost (assumption A1 below), the set $G_\phi(x)$ will be reduced to a single point for Lebesgue almost every $x$. Note also that when $M$ is compact (which we will assume throughout the remainder of the paper), the set $G_\phi(x)$ is never empty when $\phi$ is $c$-convex.

Brenier’s result was generalized in a natural way to Riemannian manifolds by McCann: Let $M$ be a manifold, with Riemannian metric $g$, compact and without boundary, with distance function $d(\cdot, \cdot)$. For $u, v \in T_x(M)$, $(u, v)_g(x)$ (or in short $(u, v)_g$) denotes the scalar product on $T_x(M)$ with respect to the metric $g$, $|v|^2_g = (v, v)_g$. From the results of [18], in the case where $c = d^2/2$, the optimal map can be expressed as a so-called gradient map, that is

$$G_\phi(x) = \exp_x(\nabla_g \phi(x)),$$

where $\nabla_g$ denotes the gradient with respect to the Riemannian metric $g$ on $M$ (from now on, we omit the subscript $g$), and $\phi$ is some $c$-convex potential. For a general cost, one needs first to introduce the $c$-exponential map $c\text{-exp}_x(\cdot)$, defined as the