SBV Regularity for Hamilton–Jacobi Equations in $\mathbb{R}^n$

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Abstract

In this paper we study the regularity of viscosity solutions to the following Hamilton–Jacobi equations

$$\partial_t u + H(D_x u) = 0 \quad \text{in } \Omega \subset \mathbb{R} \times \mathbb{R}^n.$$ 

In particular, under the assumption that the Hamiltonian $H \in C^2(\mathbb{R}^n)$ is uniformly convex, we prove that $D_x u$ and $\partial_t u$ belong to the class $SBV_{loc}(\Omega)$.

1. Introduction

In this paper, we consider viscosity solutions $u$ to Hamilton–Jacobi equations

$$\partial_t u + H(D^2 u) = 0 \quad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n.$$ 

(1)

As is well known, solutions of the Cauchy problem for (1) develop singularities of the gradient in finite time, even if the initial data $u(0, \cdot)$ are extremely regular. The theory of viscosity solutions, introduced by Crandall and Lions 30 years ago, provides several powerful existence and uniqueness results which allow one to go beyond the formation of singularities. Moreover, viscosity solutions are the limit of several smooth approximations of (1). For a review of the concept of viscosity solution and the related theory for equations of type (1) we refer to [4,5,10].

In this paper we are concerned about the regularity of such solutions, under the following key assumption:

$$H \in C^2(\mathbb{R}^n) \quad \text{and} \quad c_H^{-1} I d_n \leq D^2 H \leq c_H I d_n \quad \text{for some } c_H > 0. \quad (2)$$

There is a vast literature about this issue. As is well-known, under the assumption (2), any viscosity solution $u$ of (1) is locally semiconcave in $x$. More precisely, for every $K \subset \subset \Omega$ there is a constant $C$ (depending on $K$, $\Omega$ and $c_H$) such that the
function $x \mapsto u(t, x) - C|x|^2$ is concave on $K$. This easily implies that $u$ is locally Lipschitz and that $\nabla u$ has locally bounded variation, that is, that the distributional Hessian $D^2_x u$ is a symmetric matrix of Radon measures. It is not difficult, then, to see that the same conclusion holds for $\partial_t D_x u$ and $\partial_{tt} u$. Note that this result is independent of the boundary values of $u$ and can be regarded as an interior regularization effect of the equation.

The rough intuitive picture that one has in mind is, therefore, that of functions which are Lipschitz and whose gradient is piecewise smooth, undergoing jump discontinuities along a set of codimension 1 (in space and time). A refined regularity theory, which confirms this picture and goes beyond, analyzing the behavior of the functions where singularities are formed, is available under further assumptions on the boundary values of $u$ (we refer to the book [5] for an account on this research topic). However, if the boundary values are just Lipschitz, these results do not apply and the corresponding viscosity solutions might be, indeed, quite rough, if we understand their regularity only in a pointwise sense.

In this paper we prove that the BV regularization effect is, in fact, more subtle and there is a measure-theoretic analog of “piecewise $C^1$ with jumps of the gradients”. As a consequence of our analysis, we know, for instance, that the singular parts of the Radon measures $\partial_{x_j} u$, $\partial_{x_i} u$ and $\partial_t u$ are concentrated on a rectifiable set of codimension 1. This set is, indeed, the measure theoretic jump set $J_{D_x u}$ of $D_x u$ (see below for the precise definition). This excludes, for instance, that the second derivative of $u$ can have a complicated fractal behaviour. Using the language introduced in [7] we say that $D_x u$ and $\partial_t u$ are (locally) special functions of bounded variation, that is, they belong to the space $SBV_{loc}$ (we refer to the monograph [2] for more details). A typical example of a 1-dimensional function which belongs to BV but not to SBV is the classical Cantor staircase (compare with Example 1.67 of [2]).

**Theorem 1.1.** Let $u$ be a viscosity solution of (1), assume (2) and set $\Omega_t := \{x \in \mathbb{R}^n : (t, x) \in \Omega\}$. Then, the set of times

$$S := \{t : D_x u(t, .) \notin SBV_{loc}(\Omega_t)\}$$

is, at most, countable. In particular $D_x u$, $\partial_t u \in SBV_{loc}(\Omega)$.

**Corollary 1.2.** Under assumption (2), the gradient of any viscosity solution $u$ of

$$H(D_x u) = 0 \text{ in } \Omega \subset \mathbb{R}^n,$$

belongs to $SBV_{loc}(\Omega)$.

Theorem 1.1 was first proved by Luigi Ambrosio and the second author in the special case $n = 1$ (see [3] and also [11] for the extension to Hamiltonians $H$ depending on $(t, x)$ and $u$). Some of the ideas of our proof originate in the work [3]. However, in order to handle the higher dimensional case, some new ideas are needed. In particular, a key role is played by the geometrical theory of monotone functions developed by Alberti and Ambrosio in [1].