

**Abstract**

We consider one-component lattice gases with local dynamics and a stationary product Bernoulli measure on $\mathbb{Z}^d$. We study the scaling exponents of the space-time correlations of the system in equilibrium at a given density. We consider a variance-like quantity computed from the correlations called the diffusivity (connected to the Green–Kubo formula) and give rigorous upper and lower bounds on it that depend on the dimension and the local behavior of the macroscopic flux function. Our results identify the cases in which the system scales superdiffusively; these cases have been predicted before, using non-rigorous scaling arguments. Our main tool is the resolvent method: the estimates are the result of a careful analysis of a complicated variational problem.

1. Introduction

We study the scaling properties of a family of lattice gas models on the $d$-dimensional lattice. The model describes particles jumping on $\mathbb{Z}^d$ with an exclusion rule (that is, no multiple particles per site) and a Markovian local evolution which is translation invariant. This means that each particle can jump to another site in its finite neighborhood (if that site is empty) with a rate that depends only on the configuration of particles observed in that finite neighborhood. These models are sometimes called speed change models; the simplest example, when the jump rates depend only on the size of the jump, is called the exclusion process. The models we consider will have the additional special property that the product Bernoulli measures with a fixed parameter $\rho$ will be stationary for the evolution; this will impose an additional divergence-like condition on the rate functions (see (16)).

It has been observed [32] that interacting particle systems or growth models can exhibit superdiffusive behavior: certain physically relevant quantities scale differently (with a higher exponent) than one would expect from the central limit
In [2] non-rigorous physical arguments were used to show that the space-time correlations in a family of one-dimensional lattice gases at a fixed equilibrium density scale superdiffusively, with an exponent of $2/3$ instead of the $1/2$ expected from the central limit theorem. In fact, it was predicted that the appearance of this exponent is universal in one dimension, in the sense that the same exponent shows up as long as the so-called macroscopic flux function has a non-vanishing second derivative at the considered equilibrium density. Analogous predictions can be made in other dimensions, or with different assumptions, on the local behaviour of the macroscopic flux. The other superdiffusive cases are the one-dimensional models with a simple inflection point at the equilibrium or two-dimensional models with non-vanishing second derivatives at the equilibrium. In both of these cases the scaling is predicted to be logarithmically superdiffusive, that is, one needs to scale with $t^{1/2}(\log t)^{\xi}$ to obtain a nontrivial scaling limit.

In recent years some of the scaling predictions have been derived rigorously for a handful of special models, notably the polynuclear growth model and asymmetric exclusion in $d = 1$ (see [9] for a survey) and asymmetric exclusion in $d = 2$ [36]. But we are still far from proving the predicted universality, and there is no rigorous result of any sort for the one-dimensional case at an inflection point. Our goal in the current paper is to study the scaling exponents in a suitably general family of models, to give superdiffusive estimates in all the cases when the model is expected to be superdiffusive and to give diffusive estimates in all the cases when the model is expected to be diffusive. We will study the scaling exponents by considering a variance-like quantity called diffusivity computed from the equilibrium space-time corrections. Our results give rigorous bounds on the Laplace transform of that diffusivity which identify the superdiffusive and diffusive cases as predicted by the physical scaling arguments. Using the Green–Kubo formula, which relates the diffusivity to a certain space-time average of the flux-flux correlations, we can express the Laplace transform of the diffusivity as the $H_{-1}$ norm of a local function, which leads to a (complicated) variational formula. In order to get the bounds on the diffusivity we need to understand this variational problem and get upper and lower bounds on its solution.

### 1.1. Motivation and Heuristic Scaling Predictions

In order to explain the predicted scalings, we use as a proxy the formal continuum version of the model, which is the stochastically forced diffusion equation with non-linear drift in $\mathbb{R}^d$,

$$
\partial_t u = \nabla \cdot w(u) + \nabla \cdot D \nabla u + \sqrt{D} \nabla \cdot \xi, \quad (1)
$$

where the current $w(u)$ is a polynomial, $D$ is a diffusivity and $\xi_i$, $i = 1, \ldots, d$ are independent space-time white noises. In one dimension only, (1) coincides with the stochastic Hamilton-Jacobi equation for a height function $h$ defined through $\nabla h = u$,

$$
\partial_t h = w(\nabla h) + D \Delta h + \sqrt{D} \xi. \quad (2)
$$