Periodic Motions of Stokes and Navier–Stokes Flows Around a Rotating Obstacle

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Abstract

We prove the existence and uniqueness of periodic motions to Stokes and Navier–Stokes flows around a rotating obstacle \( D \subset \mathbb{R}^3 \) with the complement \( \Omega = \mathbb{R}^3 \setminus D \) being an exterior domain. In our strategy, we show the \( C_b \)-regularity in time for the mild solutions to linearized equations in the Lorentz space \( L^{3,\infty}(\Omega) \) (known as weak-\( L^3 \) spaces) and prove a Massera-typed Theorem on the existence and uniqueness of periodic mild solutions to the linearized equations in weak-\( L^3 \) spaces. We then use the obtained results for such equations and the fixed point argument to prove such results for Navier–Stokes equations around a rotating obstacle. We also show the stability of such periodic solutions.

1. Introduction and Preliminaries

We consider the Navier–Stokes flows around a rotating obstacle in \( \mathbb{R}^3 \) described by the following equations:

\[
\begin{align*}
 u_t + (u \cdot \nabla) u - \Delta u + \nabla p &= (\zeta \times x) \cdot \nabla u - \zeta \times u + \text{div} F &\quad &\text{in } \Omega \times (0, \infty), \\
 \nabla \cdot u &= 0 &\quad &\text{in } \Omega \times (0, \infty), \\
 u &= \zeta \times x &\quad &\text{on } \partial \Omega \times (0, \infty), \\
 u|_{t=0} &= u_0 &\quad &\text{in } \Omega, \\
 \lim_{|x| \to \infty} u &= 0
\end{align*}
\]

(1.1)

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where \( u = u(x, t) \) is the velocity field of the liquid; \( p = p(x, t) \) is the pressure field; \( \text{div} F \) represents the external force with \( F = F(x, t) \) being a second-order tensor field; \( \zeta = (0, 0, \alpha) \) is a constant vector representing angular velocity of the rotation around the \( x_3 \)-axis of an obstacle \( D \subset \mathbb{R}^3 \) with complement \( \Omega = \mathbb{R}^3 \setminus D \) being an exterior domain.

After a first contribution in the mid 1990s by Salvi [21] and Maremonti and Padula [14], a systematic study of the existence of time-periodic Navier–Stokes flows around an obstacle started only several years later thanks to the deep work of Yamazaki [26], and has then been successively continued by a number of authors (see [5–7, 11, 23, 25]). As is already known, when considering Stokes and Navier–Stokes flows around an obstacle (at rest or rotating), one faces the difficulty that the Stokes semigroup is not exponentially stable. Therefore, in that case, it is difficult to apply some direct and simple criteria for the existence and uniqueness of periodic solutions to evolution equations as in, for example, [1, 16, 22, 27] and references therein. Thus, several new approaches have been introduced to overcome this difficulty.

For an obstacle at rest, the first approach to study the periodic motions of Navier–Stokes flows in exterior domains has been introduced by Maremonti and Padula [14] based on the “invading domain” technique (introduced by Heywood in [9]) combined with the method of Prodi–Prouse–Yudovich in [19, 20, 28] (which can be considered as an excellent combination between Poincaré mapping and Faedo-Galerkin methods). The second approach, introduced by Galdi and Sohr [5], extended Serrin’s method (which related the stability of solutions with the existence of a periodic one) and combined with function spaces featuring the decay of the solutions at spatial infinity. The last approach that we would like to mention was given by Yamazaki [26], and exploited the interpolation features of the weak-\( L^3 \) spaces and Kato’s iteration technique [12] to prove the existence and uniqueness of periodic mild solutions to Navier–Stokes equations on exterior domains.

For a moving obstacle, in [6], Galdi and Silvestre assumed a prescribed motion of the body given by a time-periodic translation velocity and time-periodic angular velocity and used a combination of the “invading domain” technique and the Prodi–Prouse–Yudovich method to show the existence (in \( L^2 \)-spaces) of a weak time-periodic flow around a moving body. The existence of strong periodic solutions was also given in [6] under some small conditions. The uniqueness and the stability of such solutions were left unsolved in their paper.

It is interesting that we can invoke the folklore result by Massera [15] for periodic solutions to ordinary differential equations (which roughly said that if an ODE has a bounded solution then it has a periodic one) to study the existence and uniqueness of periodic motions of Stokes flows around a rotating obstacle. We would like to note that the result and method of Massera have been extended to various types of equations in Banach spaces, and we refer the reader to Zubelevich [29] for a survey on the state of the art and a nice generalization of such Massera-type theorems. Moreover, in that paper, Zubelevich introduced a mean-ergodic-theorem approach to generalize the result of Massera.

Concretely, in the present paper, we combine the above-mentioned approach introduced by Yamazaki [26], the folklore result given by Massera [15] and the