Landau Damping in Sobolev Spaces for the Vlasov-HMF Model

ERWAN FAOU & FRÉDÉRIC ROUSSET

Communicated by C. DAFERmos

Abstract

We consider the Vlasov-HMF (Hamiltonian Mean-Field) model. We consider solutions starting in a small Sobolev neighborhood of a spatially homogeneous state satisfying a linearized stability criterion (Penrose criterion). We prove that these solutions exhibit a scattering behavior to a modified state, which implies a nonlinear Landau damping effect with polynomial rate of damping.

1. Introduction

In this paper we consider the Vlasov-HMF model. This model has received much interest in the physics literature for many reasons: It is a simple ideal toy model that keeps several features of the long range interactions, it is a simplification of physical systems like charged or gravitational sheet models and it is rather easy to make numerical simulations on it. We refer for example to [1,2,6,7,13] for more details.

We shall study the long time behavior of solutions to this model for initial data that are small perturbations in a weighted Sobolev space to a spatially homogeneous stationary state satisfying a Penrose type stability condition. We shall prove that the solution scatters when times goes to infinity towards a modified state close to the initial data in a Sobolev space of lower order. This result implies a nonlinear Landau damping effect with polynomial rate for the solution which converges weakly towards a modified spatially homogeneous state. In the case of analytic or Gevrey regularity, this result has been shown to hold for a large class of Vlasov equations that contains the Vlasov–Poisson system by MOUHOT and VILLANI [12] (see also the recent simplified proof [4]). Some earlier partial results were obtained in [5,8].

The related problem of the stability of the Couette flow in the two-dimensional Euler equation has been also studied recently [3]. The question left open in these papers is the possibility of nonlinear Landau damping for Sobolev perturbations.
In this case, one cannot hope for an exponential damping, but we can wonder if it could occur by allowing polynomial rates. For the Vlasov–Poisson system, it was proven in [10] that this is false with rough Sobolev regularity due to the presence of arbitrarily close travelling BGK states. Nevertheless, this obstruction disappears for sufficiently high Sobolev regularity as also proven in [10]. These arguments can also be extended to a large class of Vlasov equations and in particular the HMF model. Note that the regularity of the interaction kernel does not play an essential part in the argument of [12] (though the decay in Fourier space provided by the Coulomb interaction seems critical), it is more the nonlinear “plasma-echo” effect which is crucial to handle. Besides the physical interest of the HMF model, it is thus mathematically interesting to study the possibility of nonlinear Landau-damping in Sobolev spaces for this simple model.

1.1. The Vlasov-HMF Model

The Vlasov-HMF model reads

\[
\partial_t f(t, x, v) + v \partial_x f(t, x, v) = \partial_x \left( \int_{\mathbb{R} \times \mathbb{T}} P(x - y) f(t, y, u) \, du \, dy \right) \partial_v f(t, x, v),
\]

(1.1)

where \((x, v) \in \mathbb{T} \times \mathbb{R}\) and the kernel \(P(x)\) is given by \(P(x) = \cos(x)\). Note that the main difference with the Vlasov–Poisson equation is the regularity of the kernel: in this latter case, \(P(x) = \sum_{k \geq 0} k^{-2} \cos(kx)\) is the kernel associated with the inverse of the Laplace operator. The HMF model is thus the simplest nonlinear model with the structure (1.1). We consider initial data under the form \(f_0(x, v) = \eta(v) + \epsilon r_0(x, v)\) where \(\epsilon\) is a small parameter and \(r_0\) is of size one (in a suitable functional space). This means that we study small perturbations of a stationary solution \(\eta(v)\). We shall thus write the solution at time \(t\) under the form

\[
f(t, x, v) = \eta(v) + \epsilon r(t, x, v).
\]

We are interested in the study of the behavior of \(f\) when time goes to infinity. To filter the effect of the free transport, it is convenient to introduce (as in [4,12]) the unknown \(g(t, x, v) = r(t, x + tv, v)\) that is solution of the equation

\[
\partial_t g = \{\phi(t, g), \eta\} + \epsilon \{\phi(t, g), g\}.
\]

(1.2)

where

\[
\phi(t, g)(x, v) = \int_{\mathbb{R} \times \mathbb{T}} (\cos(x - y + t(v - u))) g(t, y, u) \, du \, dy
\]

(1.3)

and \(\{f, g\} = \partial_x f \partial_v g - \partial_v f \partial_x g\) is the usual microcanonical Poisson bracket. We shall usually write \(\phi(t)\) when the dependence in \(g\) is clear.

We shall work in the following weighted Sobolev spaces, for \(m_0 > 1/2\) be given, we set

\[
\|f\|_{\mathcal{H}^n}^2 = \sum_{|p| + |q| \leq n} \int_{\mathbb{T} \times \mathbb{R}} (1 + |v|^2)^{m_0} |\partial_x^p \partial_v^q f|^2 \, dx \, dv,
\]

(1.4)