A Minimal Integrity Basis for the Elasticity Tensor

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Abstract

We definitively solve the old problem of finding a minimal integrity basis of polynomial invariants of the fourth-order elasticity tensor $C$. Decomposing $C$ into its $SO(3)$-irreducible components we reduce this problem to finding joint invariants of a triplet $(a, b, D)$, where $a$ and $b$ are second-order harmonic tensors, and $D$ is a fourth-order harmonic tensor. Combining theorems of classical invariant theory and formal computations, a minimal integrity basis of 297 polynomial invariants for the elasticity tensor is obtained for the first time.

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1. Introduction

In solids mechanics when the matter is slightly deformed, the local state of strain is modelled, at each material point, by a second-order symmetric tensor $\varepsilon$. 
The local stress resulting from the imposed strain is classically described by another second-order symmetric tensor, the Cauchy stress $\sigma$. The way stress and strain are related is defined by a constitutive law. According to the intensity of strain, the nature of the material, and external factors such as the temperature, the nature and type of constitutive laws can vary widely [38,55].

Among constitutive laws, linear elasticity is one of the simplest to model. It supposes a linear relationship between the strain and the stress tensor at each material point, $\sigma = C : \varepsilon$, in which $C$ is a fourth-order tensor, and an element of a 21-dimensional vector space $E_{\text{Ela}}$ [3,27,35,46]. From a physical point of view, this relation, which is the 3D extension of Hooke’s law for a linear spring, $F = k \Delta x$, encodes the elastic properties of a body in the small perturbation hypothesis [74].

Due to the existence of a micro-structure at a scale below the one used for the continuum description, elastic properties of many homogeneous materials are anisotropic, i.e. they vary with material directions. Elastic anisotropy is very common and can be encountered in natural materials (rocks, bones, crystals, …) as well as in manufactured ones (composites, textiles, extruded or rolled irons, …) [3,17,28]. Measuring and modelling the elastic anisotropy of materials is of critical importance for many applications, ranging from the anisotropic fatigue of forged steel [68], the damaging of materials [34,39], to the study of wave propagation in complex materials such as bones [4,72] or rocks [8,49]. More recently, the development of acoustic and elastic meta-materials and the wish to conceive paradoxical materials gave a new impulse for the study of anisotropic elasticity [3,48,61,71].

Working with elastic materials implies the need to identify and distinguish them. A natural question is: “How do we give different names to different elastic materials?” Despite its apparent simplicity, this question formulated for 3D elastic media is a rather hard problem to solve. An elasticity tensor $C$ represents a homogeneous material in a specific orientation with respect to a fixed frame and a rotation of the body results in another elasticity tensor $\tilde{C}$ representing the same material. Each homogeneous material is characterized by many elasticity tensors and coordinate-based designation clearly cannot label elastic materials uniquely.

From a mathematical point of view, the material change of orientation makes $C$ move in $E_{\text{Ela}}$. Classifying anisotropic materials amounts to describing the orbits of the action of the rotation group $SO(3, \mathbb{R})$ on $E_{\text{Ela}}$. This can be achieved by determining a finite system of invariants which separates the orbits.

The analogous problem in plane elasticity for the elasticity tensor in bi-dimensional space under the action of the orthogonal group $O(2)$ has already been solved by numerous authors [7,14,37,44,89,90]).

The problem in 3D is much more complicated. The first attempt to define such intrinsic parameters goes back to the seminal work of Lord Kelvin [87], was rediscovered later by Rychlewski [73] and followed since then by many authors [12,20,60,67,97]. It is based on the representation of the elasticity tensor as a symmetric second-order tensor in $\mathbb{R}^6$ and the use of its spectral decomposition. However, even if the six eigenvalues of this second-order tensor are invariants, they do not separate the orbits. Worse, the geometry of the problem, which is based on the group $SO(3, \mathbb{R})$ and not $SO(6, \mathbb{R})$, is lost.