Quasihyperbolic boundary conditions and Poincaré domains

Pekka Koskela · Jani Onninen · Jeremy T. Tyson

Received: 2 May 2000 / Published online: 17 June 2002 – © Springer-Verlag 2002

Abstract. We prove that a domain in $\mathbb{R}^n$ whose quasihyperbolic metric satisfies a logarithmic growth condition with coefficient $\beta \leq 1$ is a $(q, p)$-Poincaré domain for all $p$ and $q$ satisfying $p \in [1, \infty) \cap (n - n\beta, n)$ and $q \in [p, \beta p^*)$, where $p^* = np/(n - p)$ denotes the Sobolev conjugate exponent. An elementary example shows that the given ranges for $p$ and $q$ are sharp. The proof makes use of estimates for a variational capacity. When $p = 2$ we give an application to the solvability of the Neumann problem on domains with irregular boundaries. We also discuss the relationship between this growth condition on the quasihyperbolic metric and the $s$-John condition.

Mathematics Subject Classification (2000): 46E35, 30F45, 31B15, 28A78

1 Introduction

Let $\Omega$ be a domain of finite volume in $\mathbb{R}^n$, $n \geq 2$, and let $1 \leq p \leq q < \infty$. We say that $\Omega$ is a $(q, p)$-Poincaré domain if there exists a constant $M_{q, p} = M_{q, p}(\Omega)$ so that

$$
\left( \int_{\Omega} |u(x) - u_\Omega|^q \, dx \right)^{1/q} \leq M_{q, p} \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{1/p}
$$

for all $u \in C^\infty(\Omega)$. Here $u_\Omega = |\Omega|^{-1} \int_{\Omega} u(x) \, dx$. When $q = p$ we say that $\Omega$ is a $p$-Poincaré domain.

It is a problem of some interest to determine geometric conditions on a domain $\Omega$ (possibly with a very irregular boundary) sufficient to guarantee the satisfaction of the Poincaré inequality (1.1). A number of geometric assumptions (cone/cusp conditions, John conditions, etc.) have been considered in this context.

Pekka Koskela, Jani Onninen
Department of Mathematics, University of Jyväskylä, P.O. Box 35 (MaD), FIN-40351, Jyväskylä, Finland (e-mails: {pkoskela,jaonninen}@math.jyu.fi)

Jeremy T. Tyson
Department of Mathematics, State University of New York, Stony Brook, NY 11794-3651, USA (e-mail: tyson@math.sunysb.edu)

All three authors were supported by the Academy of Finland, project 39788. The last author was also supported by an NSF Postdoctoral Research Fellowship. The research for this paper was done while the last author was a visitor at the University of Jyväskylä. He wishes to thank the department for its hospitality.
context. Our purpose in this paper is to study sufficient conditions for the Poincaré inequality in terms of another geometric assumption, namely, a growth condition on the quasihyperbolic metric in \( \Omega \).

Let \( \Omega \subseteq \mathbb{R}^n, n \geq 2 \). The quasihyperbolic distance between a pair of points \( x, y \in \Omega \) is defined to be

\[
k_{\Omega}(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds}{\text{dist}(z, \partial \Omega)},
\]

where the infimum is taken over all curves \( \gamma \) in \( \Omega \) joining \( x \) to \( y \). This metric arises naturally in the theory of conformal geometry where, for example, it plays an important role in the study of the boundary behavior of quasiconformal maps. As another application, we mention the result of Jones [11] characterizing BMO-extension domains in terms of a growth condition on the quasihyperbolic metric. See the survey article [14] for further applications.

A connection between the quasihyperbolic metric and the global Poincaré inequality (1.1) was first demonstrated by Jerison [9], who proved that a planar domain \( \Omega \) of finite area is a 2-Poincaré domain provided

\[
\int_{\Omega} k_{\Omega}(x_0, x) \, dx < \infty
\]

for some (each) \( x_0 \in \Omega \). The analogous result in higher dimensions, due to Hurri [7] and Smith and Stegenga [23], states that a domain \( \Omega \subseteq \mathbb{R}^n \) of finite volume is an \( n \)-Poincaré domain if

\[
\int_{\Omega} k_{\Omega}(x_0, x)^{n-1} \, dx < \infty
\]

for some (each) \( x_0 \in \Omega \). A natural question then arises: can the integrability condition (1.2) be verified under some simpler geometric restriction on the quasihyperbolic metric?

Let \( \beta > 0 \). We say that \( \Omega \) satisfies a \( \beta \)-quasihyperbolic boundary condition if the growth condition

\[
k_{\Omega}(x_0, x) \leq \frac{1}{\beta} \log \frac{\text{dist}(x_0, \partial \Omega)}{\text{dist}(x, \partial \Omega)} + C_0
\]

is satisfied for all \( x \in \Omega \), where \( x_0 \in \Omega \) is a fixed basepoint and \( C_0 = C_0(x_0) < \infty \). Gehring and Martio [3] demonstrated a close connection between condition (1.3) and global regularity (specifically, Hölder continuity) of quasiconformal maps; their results generalize previous work of Becker and Pommerenke [1] for conformal maps in simply connected plane domains. By results from [23],

\[
\int_{\Omega} k_{\Omega}(x_0, x)^p \, dx < \infty
\]

for all \( p \geq 1 \) whenever \( \Omega \) satisfies (1.3) for some \( \beta > 0 \); it follows that such a domain is necessarily an \( n \)-Poincaré domain.