Hankel and Toeplitz–Schur multipliers

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Abstract. We study the problem of characterizing Hankel–Schur multipliers and Toeplitz–Schur multipliers of Schatten–von Neumann class $S_p$ for $0 < p < 1$. We obtain various sharp necessary conditions and sufficient conditions for a Hankel matrix to be a Schur multiplier of $S_p$. We also give a characterization of the Hankel–Schur multipliers of $S_p$ whose symbols have lacunary power series. Then the results on Hankel–Schur multipliers are used to obtain a characterization of the Toeplitz–Schur multipliers of $S_p$. Finally, we return to Hankel–Schur multipliers and obtain new results in the case when the symbol of the Hankel matrix is a complex measure on the unit circle.

1. Introduction

The Schur product of matrices $A = \{a_{jk}\}$ and $B = \{b_{jk}\}$ is defined as the matrix $A \ast B = \{a_{jk}b_{jk}\}$ whose entries are the products of the entries of $A$ and $B$.

If we identify in a natural way the bounded linear operators on $\ell^2$ with their matrix representation with respect to the standard orthonormal basis of $\ell^2$, we can study Schur multipliers of various classes of linear operators on $\ell^2$. Namely, if $X$ is a class of bounded linear operators on $\ell^2$, we say that a matrix $A$ is a Schur multiplier of $X$ if and only if $\forall B \in X \Rightarrow A \ast B \in X$.

We are interested in this paper in Schur multipliers of Schatten–von Neumann classes $S_p$ (see [GK], [BS1] for information on the classes $S_p$). For $0 < p < \infty$ we denote by $M_p$ the space of Schur multipliers of $S_p$ and we put $\|A\|_{M_p} = \sup \{\|A \ast B\|_{S_p} : \|B\|_{S_p} \leq 1\}$.

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It is easy to see that for $p \geq 1$ the functional $\| \cdot \|_{M^p}$ is a norm on $M^p$. Using the triangle inequality
\[ \| T_1 + T_2 \|_p \leq \| T_1 \|_p + \| T_2 \|_p, \quad 0 < p < 1, \]
(1.1)
one can easily see that
\[ \| A_1 + A_2 \|_{M^p} \leq \| A_1 \|_{M^p} + \| A_2 \|_{M^p}, \quad 0 < p < 1. \]
(1.2)
We denote by $M^p$ the class of Schur multipliers of the space $B$ of bounded linear operators on $\ell^2$.

In this paper we are going to study the Hankel–Schur multipliers of $S^p$, i.e., matrices of class $M^p$ of the form
\[ \{ \gamma_{j+k} \}_{j,k \geq 0} \]
and the Toeplitz–Schur multipliers of $S^p$, i.e., matrices of class $M^p$ of the form
\[ \{ \tau_{j-k} \}_{j,k \geq 0}. \]

Let us summarize briefly some well-known properties of classes $M^p$. The class $M^2$ is the space of matrices with bounded entries. If $1 < p < \infty$, then $M^p = M^p'$, where $1/p + 1/p' = 1$. The space $M^1$ coincides with $M_1$. This follows from the facts that the dual space to $S^p$ can be identified in a natural way with $S^p'$ and the dual to $S^1$ can be identified with $B$.

Next, interpolating between the classes $S^p$, one can easily see that $M^p_1 \subset M^p_2$ if $0 < p_1 \leq p_2 \leq 2$.

To describe the space $M$, we consider the projective tensor product $\ell^\infty \widehat{\otimes} \ell^\infty$ that consists of matrices $C = \{ c_{jk} \}_{j,k \geq 0}$ for which there exist sequences $X(n) = \{ x_j \}_{j \geq 0} \in \ell^\infty$ and $Y(n) = \{ y_j \}_{j \geq 0} \in \ell^\infty$, $n \geq 0$, such that
\[ c_{jk} = \sum_{n \geq 0} x_{n+k} y_{n-j}, \]
and
\[ \sum_{n \geq 0} \| x(n) \|_\ell^\infty \| y(n) \|_\ell^\infty < \infty. \]

The norm $\| C \|_{\ell^\infty \widehat{\otimes} \ell^\infty}$ is by definition the infimum in (1.4) over all sequences $X(n)$ and $Y(n)$ satisfying (1.3).

For any positive integer $m$ we consider the matrix $Q(m) = \{ q(m)_{jk} \}_{j,k \geq 0}$ defined by
\[ q(m)_{jk} = \begin{cases} 1, & j \leq m, k \leq m, \\ 0, & \text{otherwise}. \end{cases} \]
(1.5)
It is well known that a matrix $A$ belongs to $M^p_\infty$ if and only if
\[ \sup_m \| Q(m) \otimes A \|_{\ell^\infty \widehat{\otimes} \ell^\infty} < \infty. \]