Dolbeault Isomorphism for CR Manifolds

Christine Laurent-Thiébaut · Jürgen Leiterer

Received: 17 April 2001 / Published online: 2 December 2002 – © Springer-Verlag 2002

Mathematics Subject Classification (2000): 32F40, 32F10

Let $X$ be a complex manifold of complex dimension $n$. For $p \in \mathbb{N}$ such that $0 \leq p \leq n$, we denote by $\Omega^p_X$ the sheaf of germs of holomorphic $p$-forms on $X$. If the cohomology groups of the sheaf $\Omega^p_X$ are denoted by $H^r(X, \Omega^p_X)$ and the Dolbeault cohomology groups for $C^\infty$-smooth differential forms and for currents are respectively denoted by $H^{p,r}_\infty(X)$ and $H^{p,r}_{cur}(X)$, it follows from the Dolbeault lemma for $\partial$ and the de Rham-Weil isomorphism that for $0 \leq r \leq n$

$$H^r(X, \Omega^p_X) \cong H^{p,r}_\infty(X) \cong H^{p,r}_{cur}(X).$$

The natural map $H^{p,r}_\infty(X) \to H^{p,r}_{cur}(X)$ which is actually an isomorphism is called the Dolbeault isomorphism.

If $M$ is a $C^\infty$-smooth CR manifold, it is then natural to ask which relations exist between the cohomology groups of the sheaf of germs of $CR$ $C^\infty$-smooth $p$-forms on $M$ and the cohomology groups of the tangential Cauchy-Riemann complex for $C^\infty$-smooth differential forms and for currents. Following the method used in the complex case we may hope to get some answer when there exists a Poincaré lemma for the tangential Cauchy-Riemann operator.

Let $X$ be a complex manifold of complex dimension $n$, and let $M$ be an oriented $C^\infty$-smooth generic $CR$ submanifold of real codimension $k$ in $X$. On $M$ we consider for all $p$, $0 \leq p \leq n$, the tangential Cauchy-Riemann complexes

$$[\mathcal{E}^{p,+}](M) : 0 \to [\mathcal{E}^{p,0}](M) \xrightarrow{\bar{\partial}_k} [\mathcal{E}^{p,1}](M) \to \cdots \to [\mathcal{E}^{p,n-k}](M) \to 0$$

CHRISTINE LAURENT-THIÉBAUT
Institut Fourier, Laboratoire de Mathématiques, UMR5582 (UJF-CNRS), BP 74, 38402 St Martin d’Hères Cedex, France

JÜRGEN LEITERER
Institut für Mathematik, Humboldt-Universität zu Berlin, Rudower Chaussee 25, D-12489 Berlin, Germany

Partially supported by TMR Research Network ERBFMRXCT 98063.
of $C^\infty$-smooth differential forms and currents, whose cohomology groups are respectively denoted by $H_p^r(M)$ and $H^{p, r}_{\text{cur}}(M)$. Moreover we denote by $H^r(M, \Omega^p_M)$ the Čech cohomology groups with coefficients in the sheaf $\Omega^p_M$ of germs of CR $C^\infty$-smooth $p$-forms on $M$.

If $M$ is $q$-concave, $1 \leq q \leq n-k$, i.e. the Levi form of $M$ has at least $q$ positive eigenvalues in all directions, then, by results of Henkin [4], Nacinovich [12] and Nacinovich & Valli [13], the Poincaré lemma holds for $\partial b$ in bidegree $(p, r)$ if $1 \leq r \leq q-1$ or $n-k-q+1 \leq r \leq n-k$ and $q \geq 2$, and each CR distribution extends locally to a holomorphic function.

The main result of this paper is the following theorem.

**Theorem 0.1** Let $M$ be q-concave, $1 \leq q \leq n-k$. Additional we assume that the conormal bundle of $M$ in $X$ is trivial. Then the natural map

$$H_p^r(M) \longrightarrow H^{p, r}_{\text{cur}}(M)$$

is an isomorphism if $0 \leq r \leq q-1$ or $n-k-q+2 \leq r \leq n-k$, it is injective if $r = q$ and surjective if $r = n-k-q+1$.

For small degrees, i.e. $0 \leq r < q$, the theorem is a direct consequence of the Poincaré lemma for the tangential Cauchy-Riemann operator and of the de Rham-Weil isomorphism, as in the complex case. Therefore we have to prove it only for high degrees, i.e. $r \geq n-k-q+1$ (see Theorem 4.1).

Note that if $M$ is of hypersurface type, i.e. $k = 1$, then the surjectivity of the natural map in Theorem 0.1 was proved already by Hill and Nacinovich in [6]. Moreover, in [7], Hill and Nacinovich studied the natural map $H^r(M, \Omega^p_M) \longrightarrow H^r(M)$ and obtained analogous results on surjectivity and injectivity.

As a consequence of Theorem 0.1 and Theorem 0.1 in [10] we get the following vanishing theorem of Malgrange’s type for the cohomology of the tangential Cauchy-Riemann complex of currents.

**Corollary 0.2** Let $M$ be 1-concave CR connected and non compact. Additional we assume that the conormal bundle of $M$ in $X$ is trivial. Then for $0 \leq p \leq n$

$$H^{p, n-k}_{\text{cur}}(M) = 0.$$ 

*Note added in proof:* We have to thank the referee, who directed our attention to the fact that in our proof of Theorem 0.1 we use the hypothesis that the conormal bundle of $M$ in $X$ is trivial. We want to say moreover that we do not know whether the theorem is true without this hypothesis.