The kernel of the Rost invariant, Serre’s Conjecture II and the Hasse principle for quasi-split groups $3, 6D_4, E_6, E_7$

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Dedicated to M.S. Raghunathan on his 60th birthday

Abstract. We prove that for a simple simply connected quasi-split group of type $3, 6D_4, E_6, E_7$ defined over a perfect field $F$ of characteristic $\neq 2, 3$ the Rost invariant has trivial kernel. In certain cases we give a formula for the Rost invariant. It follows immediately from the result above that if $\text{cd } F \leq 2$ (resp. $\text{vcd } F \leq 2$) then Serre’s Conjecture II (resp. the Hasse principle) holds for such a group. For a $(\mathbb{C}_2)$-field, in particular $\mathbb{C}(x, y)$, we prove the stronger result that Serre’s Conjecture II holds for all (not necessary quasi-split) exceptional groups of type $3, 6D_4, E_6, E_7$.

1. Introduction

This paper grew out of the letters [Ch98, Ch00] where we sketched how Harder’s proof [H65, H66] of the Hasse principle for exceptional groups $3, 6D_4, E_6, E_7$ over number fields can be carried over to the case of quasi-split groups defined over a perfect field of cohomological dimension $\leq 2$ and how the same ideas can be applied to describe in particular the kernel of the Rost invariant for $2E_6$.

In this paper we give full proofs of all these results. The main ones are the following. Let $G_0$ be a quasi-split simple simply connected exceptional group of type $3, 6D_4, E_6, E_7$ defined over a perfect field $F$ of characteristic $\neq 2, 3$. Then

- the kernel of the Rost invariant of $G_0$ is trivial;
- if $\text{cd } F \leq 2$, then Serre’s Conjecture II holds for $G_0$;
- if $\text{vcd } F \leq 2$, then the Hasse principle Conjecture II holds for $G_0$;
- if index and exponent of $p$-primary algebras over $F$ coincide for $p = 2, 3$, then Serre’s Conjecture II holds for an arbitrary simple simply connected group over $F$ (not necessary quasi-split) of the same type as $G_0$.

In [G97] Gille proved that a group $G$ of inner type $E_6$ or $E_7$ defined over $F$ is $F$-split iff there exist finite field extensions $E_1, \ldots, E_s$ of $F$ splitting $G$ and such that the
\begin{align*}
g.c.d.\{[E_1:F], \ldots, [E_s:F]\} &= 1.
\end{align*}

As an easy corollary of our results we obtain that the same quasi-splitting criterion is true for outer forms of type $3.6D_4$ or $2E_6$.

Another corollary says that for any $F$-group $G$ (not necessary quasi-split) of type $3.6D_4$, $1.2E_6$, or $E_7$ there exists a chain of cyclic extensions

$$L_0 = F \subset L_1 \subset \cdots \subset L_n$$

of degrees 2, 3 such that $G$ splits over $L_n$. This statement was proved by Tits [T92]. Granting Bloch–Kato’s conjecture mod 3 in degree 3 we obtain here another proof that a group of type $E_6$, $E_7$ is split by a (solvable) Galois extension of degree $2^r3^s$.

Recall that for all classical groups and groups of type $G_2$, $F_4$ Serre’s Conjecture II and the Hasse principle Conjecture II were proved by E. Bayer–Fluckiger and R. Parimala [BP95, BP98]. J. Ferrar [Fer69] essentially proved the Hasse principle Conjecture II for inner groups $E_6$ with the Tits algebras of index $\leq 3$. Note also that our results related to Serre’s Conjecture II (items 2, 4) were obtained independently by P. Gille [G01] using different methods.

S. Garibaldi [Gar01] also proved independently the triviality of the kernel of the Rost invariant for quasi-split groups $E_6$, $E_7$. His argument is based on consideration of explicit geometric realizations of quasi-split $E_6$, $E_7$ and studying properties of algebra structures which occur in his constructions. Since items 2, 3 immediately follow from the first one, Garibaldi’s result gives another proof of Serre’s Conjecture II and the Hasse principle Conjecture II for quasi-split groups $E_6$, $E_7$.

Our approach is based on different ideas which, as we have already mentioned, come back to G. Harder. In contrast to Garibaldi’s paper we focus on studying intrinsic properties of groups splitting over small extensions of the ground field $F$. The main body of the paper are Sections 4, 5 where we study properties of groups splitting over an extension $K/F$ of degree $p = 2, 3$. In this part $F$ is an arbitrary field of characteristic $\neq 2, 3$. We show that the $F$-structure of such groups can be described completely by certain numerical invariants. In the case of quadratic extensions we follow the author proof [Cher89] of the Platonov-Margulis conjecture on the projective simplicity of groups of rational points splitting over a quadratic extension. The results obtained in these two sections allow us to give a formula for the Rost invariant of strongly inner or outer forms of type $E_6$, $E_7$ splitting over an extension of degree $p = 2, 3$ in the case where the Galois group $\text{Gal}(F^s/F)$ is a pro-$p$-group (see 5.12 and 6.2.3).

Gille’s splitting criterion [G97] for groups of inner type $E_6$, $E_7$ combined with the results described in Sections 4, 5 gives proofs of the main theorems more or less quickly. This is done in Sections 6, 7, 8.

Finally we note that the same methods together with the results of [Ch94, Ch89] prove Serre’s Conjecture II for $E_8$ in each of the following cases: