Central extensions of current groups

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Abstract. In this paper we study central extensions of the identity component $G$ of the Lie group $\mathcal{C}^\infty(M, K)$ of smooth maps from a compact manifold $M$ into a Lie group $K$ which might be infinite-dimensional. We restrict our attention to Lie algebra cocycles of the form $\omega(\xi, \eta) = [\kappa(\xi, d\eta)]$, where $\kappa : \mathfrak{t} \times \mathfrak{t} \to Y$ is a symmetric invariant bilinear map on the Lie algebra $\mathfrak{t}$ of $K$ and the values of $\omega$ lie in $\Omega^1(M, Y)/d\mathcal{C}^\infty(M, Y)$. For such cocycles we show that a corresponding central Lie group extension exists if and only if this is the case for $M = S^1$. If $K$ is finite-dimensional semisimple, this implies the existence of a universal central Lie group extension $\hat{G}$ of $G$. The groups $\text{Diff}(M)$ and $\mathcal{C}^\infty(M, K)$ act naturally on $G$ by automorphisms. We also show that these smooth actions can be lifted to smooth actions on the central extension $\hat{G}$ if it also is a central extension of the universal covering group $\tilde{G}$ of $G$.

Introduction

Let $M$ be a compact manifold and $K$ a Lie group (which may be infinite-dimensional). Then the so called current groups $\mathcal{C}^\infty(M, K)$ with pointwise multiplication are interesting infinite-dimensional Lie groups arising in many circumstances. The most studied class of such groups are the loop groups ($M = S^1$ and $K$ compact) which is completely covered by Pressley and Segal’s monograph [PS86]. The goal of this paper is a systematic understanding of a certain class of central extensions of the identity components of these groups, namely those whose Lie algebra cocycle is of product type, which is defined in more detail below. Here the main point is to see which Lie algebra cocycle can be integrated to a central Lie group extension. These central extensions occur naturally in mathematical physics, where the problem to integrate projective representations of groups to representations of central extensions is at the heart of quantum mechanics ([Mic87], [LMNS98], [Wu01]). The central extensions of current groups are often constructed via representations by pulling back central extensions of certain operator groups ([Mic89]). It is our philosophy that one should try to understand the central extensions of a Lie group $G$ first, and then try to construct...
representations of these central extensions. In this context certain discreteness conditions for Lie algebra cocycles appear naturally because they ensure that the corresponding central Lie algebra extensions integrate to group representations ([Ne02b]). We think of these discreteness conditions as an abstract version of the discreteness of quantum numbers in quantum physics. As an outcome of our analysis, we will see that we do not have to impose any conditions on the group $K$ for our general results.

We now describe our results in some more detail. Let $M$ be a compact manifold, $Y$ a sequentially complete locally convex space, $\Omega^p(M,Y)$ the space of smooth $Y$-valued $p$-forms on $M$, and $\mathfrak{z}_M(Y) = \Omega^1(M,Y)/\mathfrak{d}C^\infty(M,Y)$. Then $\mathfrak{z}_M(Y)$ carries a natural locally convex topology and if $Y$ is Fréchet, then the same holds for $\mathfrak{z}_M(Y)$. Now let $K$ be a possibly infinite-dimensional connected Lie group and $\mathfrak{k}$ its Lie algebra. We associate to each invariant continuous bilinear form $\kappa: \mathfrak{k} \times \mathfrak{k} \to Y$ a continuous Lie algebra cocycle on $\mathfrak{g} := C^\infty(M,\mathfrak{k})$ by $\omega(\xi,\eta) := [\kappa(\xi,\mathfrak{d}\eta)] \in \mathfrak{z}_M(Y)$. We call such cocycles of product type. The main objective of this paper is to understand central Lie group extensions of the identity component $G := C^\infty(M,K)$ of the Lie group $C^\infty(M,K)$ corresponding to the Lie algebra cocycle $\omega$. According to the results in [Ne02b, Sect. 7], there are two obstructions for the existence of a central Lie group extension $\hat{G}$ of $G$ corresponding to $\omega$. First the image of the associated period map $\text{per}_\omega: \pi_2(G) \to \mathfrak{z}_M(Y)$ may not be discrete, and second, the adjoint action of $\mathfrak{g}$ on the Lie algebra $\widehat{\mathfrak{g}} := \mathfrak{g} \oplus_{\omega} \mathfrak{z}_M(Y)$ does not integrate to a smooth representation of $G$. The main point in the choice of this general setting is that it permits us to use arbitrary infinite-dimensional Lie groups $K$, hence in particular groups of the type $K = C^\infty(N,H)$, $H$ a finite-dimensional Lie group. Then $C^\infty(M,K) \cong C^\infty(M \times N,H)$, so that we may use product decompositions of manifolds to study current groups on manifolds.

In the first section we investigate the discreteness of the period group $\Pi_\omega := \text{im}(\text{per}_\omega)$. Our main result states that $\Pi_\omega$ is discrete for all compact manifolds $M$ if and only if it is discrete for the manifold $M = S^1$. This is remarkable because the group $\pi_2(G)$ is not well accessible for $\dim M > 2$. In Section II we turn to the case where $K$ is finite-dimensional and $\kappa: \mathfrak{t} \times \mathfrak{t} \to V(\mathfrak{t})$ is the universal invariant symmetric bilinear form on $\mathfrak{t}$. In this case we show that the period group is discrete for $M = S^1$, hence also for arbitrary $M$ by the results of Section I.

In Section III we turn to the central Lie group extensions. Here we show in particular that for any Lie algebra cocycle $\omega$ of product type the adjoint representation of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}$ integrates to a smooth Lie group representation of the generally non-connected group $C^\infty(M,K)$. Therefore the second obstruction to the existence of a central Lie group extension is always trivial, and we obtain for each $\kappa$ for which the period group $\Pi_\omega$ is discrete a central Lie group extension of the identity component $G = C^\infty(M,K)$, $\omega$. In Section IV we show that if $K$ is finite-