New properties of small Lebesgue spaces and their applications

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Received: 25 January 2002 / Revised version: 26 November 2002 / Published online: 8 April 2003 – © Springer-Verlag 2003

Abstract. We prove some new properties of the small Lebesgue spaces introduced by Fiorenza in [F]. Combining these properties with the Poincaré-Sobolev inequalities for the relative rearrangement (see [R1]), we derive some new and precise estimates either for small Lebesgue-Sobolev spaces or for quasilinear equations with data in the small Lebesgue spaces.

1. Introduction

In this note, announced in [FR], we shall give some new properties of the small Lebesgue spaces introduced by Fiorenza ([F]), denoted by $L^p(\Omega_1)$ for a bounded set $\Omega_1$, $1 < p < +\infty$. This set is smaller than the Lebesgue space $L^p(\Omega)$ and contains all $L^{p+\varepsilon}(\Omega)$ for all $\varepsilon > 0$.

If we denote by $g^*(s) = \frac{1}{s} \int_0^s |g|^*(t) dt$, $s \in (0, \text{meas}(\Omega)) = \Omega_*$, then $g^* \in L^p(\Omega_*)$ if $g \in L^p(\Omega)$ and

$$\|g\|_{L^p(\Omega)} \leq \|g^*\|_{L^p(\Omega_*)} \leq p' \|g\|_{L^p(\Omega)}.$$  \hspace{1cm} (1)

These spaces satisfy the Levi monotone convergence property. A first consequence of such a property is that if $(E_m)_{m \geq 0}$ is a monotone sequence of measurable sets with $\text{meas}(E_m) \rightarrow 0$, then for any $f \in L^p(\Omega)$, one has $\|f X_{E_m}\|_{L^p(\Omega)} \rightarrow 0$. Here $X_{E_m}$ is the characteristic function of $E_m$.

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This work has been partially performed as a part of a National Research Project supported by M.U.R.S.T. and supported by the Programma di Scambi Internazionali per la Breve Mobilità’ instituted by Universita di Napoli “Federico II”
We shall give a direct application of some of the properties of the small Lebesgue spaces, by proceeding on a direct proof of some precise pointwise estimate.

Namely, \( W^{1,N}(\Omega) \subset L^\infty(\Omega) \cap C(\Omega) \) (if \( \Omega \) is a bounded connected lipschitz set) and

\[
\text{osc}_{B(x,t)} u \leq \frac{a_N^{1/N'}}{\omega_{N-1}} \left| \Omega \right|^{1/N'} (N')^{1/N'} \left| \nabla u \right| \chi_{B(x,t)} \left| \Omega \right|^{1/N'} .
\]  (2)

(We denote by \( \cdot \) the Banach Function norm in \( L^p \). The inclusion in \( C(\Omega) \) is a consequence of the work of \( [S] \) (see also \( [DS,CP] \)) since

\[
W^{1,N}(\Omega) \subset \left\{ v \in W^{1,1}(\Omega) : \left| \nabla v \right| \in L_N^1(\Omega) \right\} .
\]

The inclusion in \( L^\infty \) is given in \( [R2] \). The main novelty in this part is the precise estimate (2).

Furthermore, if \( u \in W^{1,p}_0(\Omega) \) solution of

\[
Au = -\text{div} (\bar{a}(x,u,\nabla u)) = f \in L^{N/p}(\Omega) \quad 1 < p < N
\]

then \( u \) is bounded if \( p \leq 2 \). Furthermore if \( p < \frac{2N}{N+1} \) and \( |\Omega| = 1 \), then we have the following precise estimate:

\[
|u|_*(s) \leq c_{aNp} \left( \int_0^1 s^{\frac{N}{2p} - \frac{1}{p}} \varphi(s)^{\frac{1}{p-1}} ds \right) |g|_{N/p}(\Omega) < +\infty ,
\]  (3)

with \( \varphi(s) = \sup_{0<s<q-1} (es)^{\frac{1}{p-1}} \), \( q = \frac{N}{N-p} \), \( c_{aNp} = \frac{1}{a^p \left( \frac{N}{N-N/N} \right)^p} \quad g = |f| \).

If \( p = 2 \) then one has:

\[
|u|_*(s) \leq c_{aN2} \left( \frac{N}{N-2} \right)^2 |g|_{N/2} \]  (4)

The above estimate (4) shows, in particular, that if we consider the operator

\[ (-\Delta)^{-1} : L^{N/2}(\Omega) \rightarrow L^\infty(\Omega) \]

then we have the following estimate of the norm

\[
\|(-\Delta)^{-1}\| = \sup_{g \neq 0} \left| \frac{(-\Delta)^{-1}g}{|g|_{N/2}} \right|_{\infty} \leq \left[ \frac{1}{1 - (N-2)\omega_N \left( \frac{N}{N-N/N} \right)^p} \right]^2.
\]

For \( p > 2 \), we shall introduce the following vector space

\[
V_p = \left\{ g \in L^{N/p}_N(\Omega) : \|g\|_{N/p}^{N/p-1} \in L^{1/N}(\Omega) \right\} .
\]