Non trivial $L^q$ solutions to the Ginzburg-Landau equation

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Abstract. It is shown that the contour problem for the stationary Ginzburg-Landau equation

\begin{align}
\begin{cases}
-\Delta u = u(1 - |u|^2), & x \in \mathbb{R}^n, \quad n = 3 \text{ or } n = 4 \\
\lim_{R \to +\infty} \frac{1}{R} \int_{B_R} |\nabla u - iu\tilde{x} + 2i(2\pi i)^{\frac{n-1}{2}} f(\tilde{x}) \frac{e^{-i|x|}}{|x|^{\frac{n-1}{2}}} \tilde{x}|^2 \, dx = 0,
\end{cases}
\end{align}

where $\tilde{x} = x/r$ with $r = |x|$, is well posed in $L^4(\mathbb{R}^n)$ for a class of small data $f \in L^2(S^{n-1})$.

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1. Introduction

In 1950, V.L. Ginzburg and L.D. Landau proposed a macroscopic theory of superconductivity based on a variational functional (the free energy) associated to a scalar complex valued function $u$ and an applied magnetic field. In bounded regions the non-dimensional form of the Ginzburg-Landau free energy without magnetic potential is given by

$$
\frac{1}{2} \int_{\Omega} \left[ |\nabla u|^2 + \frac{k^2}{2} (1 - |u|^2)^2 \right] \, d\Omega,
$$

and the associated Euler-Lagrange equation is the Ginzburg-Landau equation,

$$
-\Delta u = k^2 u (1 - |u|^2), \quad \text{in } \Omega.
$$

In the physics literature, the unknown wave function $u$ represents a complex order parameter and $k$ is the so-called Ginzburg-Landau parameter, a dimensionless characteristic constant of the superconductivity material. In superconductors, of particular interest is the quantity $|u|^2$ which is proportional to the density of superconducting electrons, so that $|u| = 1$ represents a purely superconducting

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state and \(|u| = 0\) a purely normal state. For the physical background, we refer the reader to [Tin] and the many references cited in [BBH].

We observe that the dilation in the space variables: 
\[
\tilde{u}(x) = u(x/k),
\]
leads to
\[
-\Delta \tilde{u} = \tilde{u} \left(1 - |\tilde{u}|^2 \right), \quad \text{in} \quad \Omega_k = \{kx : x \in \Omega\}.
\]
As a by-product, the parameter \(k\) can be “absorbed” by using a scaling which transforms the boundary conditions on the domain \(\Omega\) into contour conditions at infinity for the function \(\tilde{u}(x)\) as \(k \to +\infty\).

Our analysis deals with the study of the limiting case \((k \to +\infty)\) when the order parameter, \(u\), is a solution of the Ginzburg-Landau equation
\[
-\Delta u = u \left(1 - |u|^2 \right), \quad \text{in} \quad \mathbb{R}^n.
\]
Traditionally, the conditions imposed on the solutions of this equation require the order parameter \(u\) to satisfy \(|u(x)| \to 1\) as \(|x| \to +\infty\) in some sense (see [BBH], [BMR], [HeHe] and [CET]). However, here we will consider a limiting condition at infinity associated to the Ginzburg-Landau equation which requires a certain decay of the solutions at infinity. In particular, we will concentrate our attention on analyzing the existence of solutions in \(L^q\) spaces. As far as we know there are no existing results of this nature.

More precisely, the present work focusses on the boundary problem (abbreviated hereafter as BP) stated as follows: Given \(f \in L^2(S^{n-1})\) find \(u \in L^q(\mathbb{R}^n)\) solution of the equation
\[
-\Delta u = u \left(1 - |u|^2 \right), \quad \text{or equivalently} \quad \Delta u + u = |u|^2 u, \quad x \in \mathbb{R}^n \tag{1}
\]
satisfying the limiting condition
\[
\lim_{R \to +\infty} \frac{1}{R} \int_{B_R} |\nabla u - iu\hat{x} + 2i(2\pi i) \frac{n-1}{2^n} f(\hat{x}) \frac{e^{-i|x|}}{|x|^{n-1}}|^2 d\hat{x} = 0, \tag{2}
\]
where \(\hat{x} = x/r\) with \(r = |x|\).

The main purpose of this paper is to establish an existence and uniqueness theory in \(L^4(\mathbb{R}^n)\) for the BP (1)–(2) with “small” data \(f \in L^2(S^{n-1})\). Our main result here is

**Theorem 1.** Assume \(n = 3\) or \(n = 4\). Let \(f \in L^2(S^{n-1})\) with \(\|f\|_{L^2(S^{n-1})} \leq \varepsilon\) (\(\varepsilon\) sufficiently small). Then there exists \(a = a(\|f\|_{L^2(S^{n-1})})\) such that the problem (1)–(2) has a unique solution \(u(\cdot)\) satisfying
\[
u \in B_a(L^4(\mathbb{R}^n)) \equiv \{u : \mathbb{R}^n \to \mathbb{C} / \|u\|_{L^4(\mathbb{R}^n)} < a\}
\]
and \(u \in C^\infty(\mathbb{R}^n) \cap \mathcal{Y}\), where
\[
\mathcal{Y} \equiv \{u : \mathbb{R}^n \to \mathbb{C} / \sup_{R \geq 1} \left(\frac{1}{R} \int_{B_R} |\nabla u|^2 + |u|^2 dx\right)^{1/2} \leq \infty\}.
\]