Differentiable perturbation of unbounded operators

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Abstract. If $A(t)$ is a $C^{1,\alpha}$-curve of unbounded self-adjoint operators with compact resolvents and common domain of definition, then the eigenvalues can be parameterized $C^1$ in $t$. If $A$ is $C^\infty$ then the eigenvalues can be parameterized twice differentiably.

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Theorem. Let $t \mapsto A(t)$ for $t \in \mathbb{R}$ be a curve of unbounded self-adjoint operators in a Hilbert space with common domain of definition and with compact resolvent.

(A) If $A(t)$ is real analytic in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized real analytically in $t$.

(B) If $A(t)$ is $C^\infty$ in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors can be parameterized smoothly in $t$, on the whole parameter domain.

(C) If $A$ is $C^\infty$, then the eigenvalues of $A(t)$ may be parameterized twice differentiably in $t$.

(D) If $A(t)$ is $C^{1,\alpha}$ for some $\alpha > 0$ in $t \in \mathbb{R}$, then the eigenvalues of $A(t)$ may be parameterized in a $C^1$ way in $t$.

Part (A) is due to Rellich [10] in 1940, see also [2] and [6], VII, 3.9. Part (B) has been proved in [1], 7.8, see also [8], 50.16, in 1997; there we gave also a different proof of (A). The purpose of this paper is to prove parts (C) and (D).

Both results cannot be improved to obtain a $C^{1,\beta}$-parameterization of the eigenvalues for some $\beta > 0$, by the first example below. In our proof of (D) the assumption $C^{1,\alpha}$ cannot be weakened to $C^1$, see the second example. For finite
dimensional Hilbert spaces part (D) has been proved under the assumption of $C^1$ by Rellich [11], with a small inaccuracy in the auxiliary theorem on p. 48: Condition (4) must be more restrictive, otherwise the induction argument on p. 50 is not valid, since the proof on p. 52 relies on the fact that all values coincide at the point in question. A proof can also be found in [6], II, 6.8. We need a strengthened version of this result, thus our proof covers it also.

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Definitions and remarks. That $A(t)$ is a real analytic, $C^\infty$, or $C^{k,\alpha}$ curve of unbounded operators means the following: There is a dense subspace $V$ of the Hilbert space $H$ such that $V$ is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$. Moreover, we require that $t \mapsto \langle A(t)u, v \rangle$ is real analytic, $C^\infty$, or $C^{k,\alpha}$ for each $u \in V$ and $v \in H$. This implies that $t \mapsto A(t)u$ is of the same class $\mathbb{R} \to H$ for each $u \in V$ by [8], 2.3 or [5], 2.6.2. This is true because $C^{k,\alpha}$ can be described by boundedness conditions only; and for these the uniform boundedness principle is valid. A function $f$ is called $C^{k,\alpha}$ if it is $k$ times differentiable and for the $k$-th derivative the expression $\frac{f^{(k)}(t) - f^{(k)}(s)}{|t-s|^\alpha}$ is locally bounded in $t \neq s$.

A sequence of continuous, real analytic, smooth, or twice differentiable functions $\lambda_i$ is said to parameterize the eigenvalues, if for each $z \in \mathbb{R}$ the cardinality $|\{i : \lambda_i(t) = z\}|$ equals the multiplicity of $z$ as eigenvalue of $A(t)$.

The proof will moreover furnish the following (stronger) versions:

(C1) If $A(t)$ is $C^{3n,\alpha}$ in $t$ and if the multiplicity of an eigenvalue never exceeds $n$, then the eigenvalues of $A$ may be parameterized twice differentiably.

(C2) If the multiplicity of any eigenvalue never exceeds $n$, and if the resolvent $(A(t) - z)^{-1}$ is $C^{3n}$ into $L(H, H)$ in $t$ and $z$ jointly, then the eigenvalues of $A(t)$ may be parameterized twice differentiably in $t$.

(C3) If the resolvent $(A(t) - z)^{-1}$ is $C^1$ into $L(H, H)$ in $t$ and $z$ jointly, then the eigenvalues of $A(t)$ may be parameterized in a $C^1$ way in $t$.

(C4) In the situations of (D) and (D1) the following holds: For any continuous parameterization $\lambda_i(t)$ of all eigenvalues of $A(t)$, each function $\lambda_i$ has right sided derivative $\lambda_i^{(+)}(t)$ and left sided one $\lambda_i^{(-)}(t)$ at each $t$, and $\{\lambda_i^{(+)}(t) : \lambda_i(t) = z\}$ equals $\{\lambda_i^{(-)}(t) : \lambda_i(t) = z\}$ with correct multiplicities.

Open problem. Construct a $C^1$-curve of unbounded self-adjoint operators with common domain and compact resolvent such that the eigenvalues cannot be arranged $C^1$.

Applications. Let $M$ be a compact manifold and let $t \mapsto g_t$ be a smooth curve of smooth Riemannian metrics on $M$. Then we get the corresponding smooth