Grope cobordism and feynman diagrams

James Conant · Peter Teichner

Received: 5 September 2002 / Revised version: 12 January 2003 / Published online: 17 October 2003 – © Springer-Verlag 2003

Abstract. We explain how the usual algebras of Feynman diagrams behave under the grope degree introduced in [CT]. We show that the Kontsevich integral rationally classifies grope cobordisms of knots in 3-space when the “class” is used to organize gropes. This implies that the grope cobordism equivalence relations are highly nontrivial in dimension 3. We also show that the class is not a useful organizing complexity in 4 dimensions since only the Arf invariant survives. In contrast, measuring gropes according to “height” does lead to very interesting 4-dimensional information [COT]. Finally, several low degree calculations are explained, in particular we show that S-equivalence is the same relation as grope cobordism based on the smallest tree with an internal vertex.

Mathematics Subject Classification (2000): 57M27

Key words or phrases: Grope cobordism, Feynman diagrams, Vassiliev invariants

Contents

1. Introduction .......................................136
2. Gropes, claspers and diagrams ..................139
   2.1. Basic notions ...................................139
   2.2. Feynman diagrams ............................140
   2.3. Maps relating capped gropes and Feynman diagrams ........142
   2.4. Claspers and Feynman diagrams ...............143
3. The grope degree ..................................149
   3.1. Feynman diagrams and the grope degree ..........149
   3.2. 4-dimensional grope cobordism: Grope concordance ....154
   3.3. Clasper moves and the grope degree ..............154
4. Low degree calculations ............................158
   4.1. The groups $\mathcal{K}/G_k$ for $k \leq 5$ ............159
   4.2. Tree types of class 4 ..........................161

J. Conant
Department of Mathematics, Cornell University, Ithaca, NY 14853-4201, USA
(e-mail: jconant@math.cornell.edu)

P. Teichner
Department of Mathematics, University of California in San Diego, 9500 Gilman Dr, La Jolla,
CA 92093-0112 (e-mail: teichner@euclid.ucsd.edu)

The first author was partially supported by NSF VIGRE grant DMS-9983660. The second author
was partially supported by NSF grant DMS-0072775 and the Max-Planck Gesellschaft.
1. Introduction

In [CT] we introduced the notion of a grope cobordism between two knots in 3-space, which places Vassiliev theory in a natural topological context. Gropes are certain 2-complexes built out of several surface stages, whose complexity can be measured by either the class (corresponding to nilpotent groups) or the height (corresponding to solvable groups). The analogy to group theory arises by observing that a continuous map $\phi$ of a circle (into some target space) represents a commutator in the fundamental group if and only if it extends to a map of a surface (which is the simplest possible grope, of class 2 and height 1). Similarly, $\phi$ represents an element in the $k$-th term of the lower central series (respectively derived series) of the fundamental group if and only if it extends to a continuous map of a grope of class $k$ (respectively height $k$).

In knot theory, one replaces continuous maps of a circle by smooth embeddings of a circle into 3-space. Accordingly, one should study embeddings of gropes into 3-space. More precisely, one obtains two sequences of new geometric equivalence relations on the set of knot types by calling two knots equivalent if they cobound an embedded grope (of a specified class or height).

It is the purpose of this paper to show that the invariants associated to grope cobordism are extremely interesting. Let $K$ be the abelian monoid of knot types, i.e. isotopy classes of oriented knots in 3-space (under connected sum). We proved in [CT] that the quotients $K/G_k := K$ modulo grope cobordism of class $k$ in 3-space, are in fact finitely generated abelian groups. In Section 3.1 we start the investigation of these groups by showing that there is an epimorphism

$$B^e_{<k} \twoheadrightarrow K/G_k,$$

where $B^e_{<k}$ is the usual (primitive) diagram space known from the theory of finite type invariants but graded by the grope degree. More precisely, $B^e_{<k}$ is the abelian group generated by connected uni-trivalent graphs of grope degree $i$, $1 < i < k$, with at least one univalent vertex and a cyclic ordering at each trivalent vertex. The relations are the usual IHX and AS relations. The grope degree is the Vassiliev degree (i.e. half the number of vertices) plus the first Betti number of the graph. Observe that both relations preserve this new degree.

We then show in Section 3.1 that as in the usual theory of finite type invariants, the above map has an inverse, the Kontsevich integral, after tensoring with the