Tight, not semi–fillable contact circle bundles

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Abstract. Extending our earlier results, we prove that certain tight contact structures on circle bundles over surfaces are not symplectically semi–fillable, thus confirming a conjecture of Ko Honda.

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1. Introduction

Let \( Y \) be a closed, oriented three–manifold. A positive, coorientable contact structure on \( Y \) is the kernel

\[
\xi = \ker \alpha \subset TY
\]

of a one–form \( \alpha \in \Omega^1(Y) \) such that \( \alpha \wedge d\alpha \) is a positive volume form on \( Y \). The pair \( (Y, \xi) \) is a contact three–manifold. In this paper we only consider positive, coorientable contact structures, so we call them simply ‘contact structures’. For an introduction to contact structures the reader is referred to [1, Chapter 8] and [7].

There are two kinds of contact structures \( \xi \) on \( Y \). If there exists an embedded disc \( D \subset Y \) tangent to \( \xi \) along its boundary, \( \xi \) is called overtwisted, otherwise it is said to be tight. The isotopy classification of overtwisted contact structures coincides with their homotopy classification as tangent two–plane fields [4]. Tight contact structures are much more mysterious, and difficult to classify. A contact structure on \( Y \) is virtually overtwisted if its pull–back to some finite cover of \( Y \) becomes overtwisted, while it is called universally tight if its pull–back to the universal cover of \( Y \) is tight.

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A contact three–manifold \((Y, \xi)\) is \textit{symplectically fillable}, or simply \textit{fillable}, if there exists a compact symplectic four–manifold \((W, \omega)\) such that (i) \(\partial W = Y\) as oriented manifolds (here \(W\) is oriented by \(\omega \wedge \omega\)) and (ii) \(\omega|_\xi \neq 0\) at every point of \(Y\). \((Y, \xi)\) is symplectically \textit{semi}–fillable if there exists a fillable contact manifold \((N, \eta)\) such that \(Y \subset N\) and \(\eta|_Y = \xi\). Semi–fillable contact structures are tight \([6,13]\). The converse is known to be false by work of Etnyre and Honda, who recently found two examples of tight but not semi–fillable contact three–manifolds \([8]\). Nevertheless, all such examples known at present are virtually overtwisted, so it is natural to wonder whether every universally tight contact structure is symplectically semi–fillable.

In this paper we study certain virtually overtwisted tight contact structures discovered by Ko Honda. Denote by \(Y_{g,n}\) the total space of an oriented \(S^1\)–bundle over \(\Sigma_g\) with Euler number \(n\). Honda gave a complete classification of the tight contact structures on \(Y_{g,n}\) \([14]\). The three–manifolds \(Y_{g,n}\) carry infinitely many tight contact structures up to diffeomorphism. The hardest part of the classification involves two virtually overtwisted contact structures \(\xi_0\) and \(\xi_1\), which exist only when \(n \geq 2g\). Honda conjectured that \(\xi_0\) and \(\xi_1\) are not symplectically semi–fillable \([14]\). The main theorem of the present paper extends our earlier results regarding these structures \([19]\), establishing Honda’s conjecture:

\textbf{Theorem 1.1.} For \(n \geq 2g > 0\), the tight contact structures \(\xi_0\) and \(\xi_1\) on \(Y_{g,n}\) are not symplectically semi–fillable.

The proof of Theorem 1.1 consists of two steps. In the first step, we derive a contact surgery presentation for \(\xi_0\) and \(\xi_1\) in the sense of \([3]\), and we use it to determine the homotopy type of \(\xi_0\) and \(\xi_1\) considered as oriented two–plane fields. This is done in Sections 2 and 3.

In the second step, using specific properties of the \(\text{Spin}^c\) structures \(t_{\xi_i}\) on \(Y_{g,n}\) induced by \(\xi_i\) \((i = 0, 1)\) we generalize a result of the first author \([17]\) so it applies to the situation at hand. Using this generalization together with an analytic computation of Nicolaescu’s \([20]\), we are able to determine the possible homotopy types of a semi–fillable contact structure inducing either \(t_{\xi_0}\) or \(t_{\xi_1}\). This is done in Section 4.

Theorem 1.1 follows immediately from the fact that the two sets of homotopy classes determined in the two steps above have empty intersection.

\section{Contact surgery presentations for \(\xi_0\) and \(\xi_1\)}

A smooth knot \(K\) in a contact three–manifold \((Y, \xi)\) which is everywhere tangent to \(\xi\) is called \textit{Legendrian}. The contact structure \(\xi\) naturally induces a framing of \(K\) called the contact framing.

Let \(\Sigma_g\) be a closed, oriented surface of genus \(g \geq 1\), and let \(\pi : Y_{g,n} \rightarrow \Sigma_g\).