Pro-\(p\)-Iwahori Hecke ring and supersingular \(\bar{\mathbb{F}}_p\)-representations

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Abstract. The motivation of this paper is the search for a Langlands correspondence modulo \(p\). We show that the pro-\(p\)-Iwahori Hecke ring \(\mathcal{H}^{(1)}\) of a split reductive \(p\)-adic group \(G\) over a local field \(F\) of finite residue field \(\mathbb{F}_q\) with \(q\) elements, admits an Iwahori-Matsumoto presentation and a Bernstein \(Z\)-basis, and we determine its centre. We prove that the ring \(\mathcal{H}^{(1)}\) is finitely generated as a module over its centre. These results are proved in [11] only for the Iwahori Hecke ring. Let \(p\) be the prime number dividing \(q\) and let \(k\) be an algebraically closed field of characteristic \(p\). A character from the centre of \(\mathcal{H}^{(1)}\) to \(k\) which is “as null as possible” will be called null. The simple \(\mathcal{H}^{(1)}_k\)-modules with a null central character are called supersingular. When \(G = \text{GL}(n)\), we show that each simple \(\mathcal{H}^{(1)}_k\)-module of dimension \(n\) containing a character of the affine subring \(\mathcal{H}^{(1)}_{aff}\) is supersingular, using the minimal expressions of Haines generalized to \(\mathcal{H}^{(1)}\), and that the number of such modules is equal to the number of irreducible \(k\)-representations of the Weil group \(W_F\) of dimension \(n\) (when the action of an uniformizer \(p_F\) in the Hecke algebra side and of the determinant of a Frobenius \(F\) in the Galois side are fixed), i.e. the number \(N_n(q)\) of unitary irreducible polynomials in \(\mathbb{F}_q[X]\) of degree \(n\). One knows that the converse is true by explicit computations when \(n = 2\) [10], and when \(n = 3\) (Rachel Ollivier).

Introduction

The results are presented before the proofs. The main results are: the conjectures 1, 2, the Iwahori-Matsumoto presentation (Theorem 1), the Bernstein basis (Theorem 2), the centre (Theorem 4), the definition 4 of supersingular, when \(G = \text{GL}(n)\) the construction of \(N_n(q)\) supersingular irreducible representations of \(\mathcal{H}^{(1)}_k\) of dimension \(n\) with a fixed action of \(p_F\) (Proposition 3, Theorem 5), and the elementary formula for \(N_n(q)\) in the appendix, reflecting the decomposition of the algebra \(\mathcal{H}^{(1)}_k\) (Proposition 4).

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1. Results

We fix a local non archimedean field $F$ (a finite extension of $\mathbb{Q}_p$ or a field of Laurent series $\mathbb{F}_q((t))$ on a finite field $\mathbb{F}_q$ with $q$ elements), of residual field $\mathbb{F}_q$ of characteristic $p$, $O_F$ the ring of integers, $\mathfrak{p}_F$ the maximal ideal, $p_F$ an uniformizer, $W_F = W(\overline{F}/F)$ a Weil group, $\mathbf{Fr}_F \in W(\overline{F}/F)$ a geometric Frobenius.\(^1\)

The classification of irreducible $\mathbb{F}_p$-representations of $GL(n, F)$ is unknown, even for the group $GL(2, F)$ when $F \neq \mathbb{Q}_p$. We replace $GL(n, F)$ by the Hecke ring $H(1) = H(GL(n, F), Iw(1))$ of the pro-$p$-radical $Iw(1)$ of an Iwahori subgroup $Iw$ of $GL(n, F)$, that we call the pro-$p$-Iwahori Hecke ring. The algebra $H(1)$ depends only on $(n, q)$. The same is true for the scalar extension $H(1)_{\mathbb{F}_p} = H(1) \otimes_{\mathbb{Z}} \mathbb{F}_p$ although $q = 0$ modulo $p$. We will sometimes denote $H(1) = H(1)(n, q)$ and $H(1)_{\mathbb{F}_p} = H(1)_{\mathbb{F}_p}(n, q)$.

Any non zero $\mathbb{F}_p$-representation of $GL(n, F)$ has a non zero $Iw(1)$-invariant vector. We hope that the functor of $Iw(1)$-invariants induces a bijection between the irreducible $\mathbb{F}_p$-representations of $GL(n, F)$ and the simple $H(1)_{\mathbb{F}_p}(n, q)$-modules. This is trivially true for $n = 1$. The supersingular $H(1)_{\mathbb{F}_p}(n, q)$-modules should be the analogues of the supercuspidal $\mathbb{F}_p$-representations of $GL(n, F)$. When $n = 1$, any $\mathbb{F}_p$-character of $H(1) \simeq \mathbb{Z}[F^*/1 + P_F]$ is supersingular.

**Definition 1.** When $n \geq 2$, a simple $H(1)_{\mathbb{F}_p}(n, q)$-module with a null central character is called supersingular.

See the precise definition 4. When $n = 2$, the functor of $Iw(1)$-invariants gives a bijection [10]
- from the irreducible NON supercuspidal smooth $\mathbb{F}_p$-representations of $GL(2, F)$ (subquotients of parabolic induced representations), to the NON supersingular irreducible $H(1)(2, q)$-modules,
- when $F = \mathbb{Q}_p$, from the irreducible supercuspidal smooth $\mathbb{F}_p$-representations of $GL(2, \mathbb{Q}_p)$, to the supersingular irreducible $H(1)(2, p)$-modules.

1.1. Numerical local Langlands correspondence modulo $p$ between $W_F$ and the pro-$p$-Iwahori Hecke ring of $GL_F$

For an integer $n \geq 2$ and for $z \in \mathbb{F}_p^*$, we denote by
- $W(n, q)$: the set of isomorphism classes of the irreducible $\mathbb{F}_p$-representations $\rho$ of $W_F$ of dimension $n$ with $\det \rho(\mathbf{Fr}_F) = z$,

\(^1\) The multiplicative group of a ring $R$ is denoted by $R^*$ and the separable algebraic closure of a field $R$ is denoted by $\overline{R}$. The field $\mathbb{F}_p$ can be replaced by any algebraically closed field $k$ of characteristic $p$. All representations of $p$-adic reductive groups will be smooth representations, and modules of Hecke algebra will be right modules.