Samuel multiplicity and Fredholm theory

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Abstract In this note we combine methods from commutative algebra and complex analytic geometry to calculate the generic values of the cohomology dimensions of a commuting multioperator on its Fredholm domain. More precisely, we prove that, for a given Fredholm tuple $T = (T_1, \ldots, T_n)$ of commuting bounded operators on a complex Banach space $X$, the limits $c_p(T) = \lim_{k \to \infty} \dim H^p(T^k, X)/k^n$ exist and calculate the generic dimension of the cohomology groups $H^p(z - T, X)$ of the Koszul complex of $T$ near $z = 0$. To deduce this result we show that the above limits coincide with the Samuel multiplicities of the stalks of the cohomology sheaves $H^p(z - T, O_{\mathbb{C}^n})$ of the associated complex of analytic sheaves at $z = 0$.

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0 Introduction and preliminaries

The Fredholm theory of single operators admits a natural extension to the case of several commuting operators. In the multivariable case the index of a Fredholm tuple $T$ is defined as the Euler characteristic of its Koszul complex. In this paper we combine methods from commutative algebra and complex analytic geometry to calculate the generic values of the cohomology dimensions of a commuting multioperator on its Fredholm domain. For single operators, the index is locally constant on the essential resolvent set $\rho_e(T)$ of $T$, while the dimensions of the kernel and cokernel of $z - T$ are only upper-semicontinuous functions in $z$. However, the discontinuity points of these functions form discrete subsets of the essential resolvent set. In the multivariable case...
the same phenomenon occurs for all the separate cohomology groups of the Koszul complex of $z - T$, except that discrete sets should be replaced by proper analytic subsets. It is the aim of this note to show that the constant values of the cohomology dimensions outside their discontinuity sets can be interpreted as suitable Hilbert-Samuel multiplicities which can be computed effectively by limit formulas well known in analogous settings of commutative algebra.

Let $T = (T_1, \ldots, T_n) \in L(X)^n$ be a commuting tuple of bounded linear operators on a complex Banach space $X$. A fundamental principle of multivariable operator theory is that all basic spectral properties of $T$ should be understood as properties of its Koszul complex. The Koszul complex $K^\bullet(z - T, X)$ is a finite complex of Banach spaces with coboundary maps

$$K^p(z - T, X) \to K^{p+1}(z - T, X), \quad \alpha s_I \mapsto \sum_{j=1}^n (z_j - T_j) x s_j \wedge s_I$$

that depend analytically on the parameter $z \in \mathbb{C}^n$. The commuting tuple $T$ is said to be invertible if the Koszul complex $K^\bullet(T, X)$ is exact. The joint spectrum $\sigma(T)$ of $T$ in the sense of J. L. Taylor consists of all points $z \in \mathbb{C}^n$ for which the tuple $z - T = (z_1 - T_1, \ldots, z_n - T_n)$ is not invertible. It is a non-empty compact subset of $\mathbb{C}^n$ which carries an analytic functional calculus, that is, there exists a continuous algebra homomorphism $O(\sigma(T)) \to L(X), \ f \mapsto f(T)$ extending the natural $O(\mathbb{C}^n)$-module structure of $X$ given by $T$ ([6] or [16]).

The commuting tuple $T$ is said to be Fredholm if all cohomology groups $H^p(T, X)$ ($p = 0, \ldots, n$) of its Koszul complex $K^\bullet(T, X)$ are finite dimensional. The Fredholm index of $T$ is defined as the Euler characteristic

$$\text{ind}(T) = \sum_{p=0}^n (-1)^p \dim H^p(T, X)$$

of its Koszul complex. The essential spectrum $\sigma_e(T)$ of $T$ consists of all points $z \in \mathbb{C}^n$ for which $z - T$ is not Fredholm. The observation that $T$ is Fredholm if and only if all cohomology sheaves of the associated complex $K^\bullet(z - T, O_{\mathbb{C}^n}^X)$ of Banach-free analytic sheaves are coherent near $0 \in \mathbb{C}^n$ allows the application of methods from complex analytic geometry. For instance, the Fredholm spectrum $\sigma(T) \cap \rho_e(T)$ is an analytic subset of the essential resolvent set $\rho_e(T) = \mathbb{C}^n \setminus \sigma_e(T)$, since it is the support of the coherent sheaf $\oplus_{p=0}^n H^p(z - T, O_{\rho_e(T)}^X)$. The discontinuity points of the functions

$$\rho_e(T) \to \mathbb{C}, \ z \mapsto \dim H^p(z - T, X) \quad (p = 0, \ldots, n)$$

form proper analytic subsets of $\rho_e(T)$. Suppose that $T$ is Fredholm. Then the stalks of the cohomology sheaves $\mathcal{H}^p = H^p(z - T, O_{\rho_e(T)}^X)$ at $0 \in \mathbb{C}^n$ are finitely generated.