Horizontal loops in Engel space

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Abstract A simple proof is given of the following result first observed by Adachi: embedded circles tangent to the standard Engel structure on \( \mathbb{R}^4 \) are classified, up to isotopy via such embeddings, by their rotation number.

Mathematics Subject Classification (2000) 57R40 · 57R42 · 57R15 · 53D10

1 Introduction

The standard Engel structure \( \mathcal{D} \) on \( \mathbb{R}^4 \) is the maximally non-integrable 2-plane distribution defined, in terms of Cartesian coordinates \((x, y, z, w)\), by the equations

\[
\begin{align*}
\,dz - y\,dx &= 0 \quad \text{and} \quad \,dw - z\,dx &= 0.
\end{align*}
\]

In other words, it is the tangent 2-plane field spanned by the vector fields

\[
e_1 := \partial_x + y\partial_z + z\partial_w \quad \text{and} \quad e_2 := \partial_y.
\]

A horizontal loop is an embedding \( \gamma : S^1 \rightarrow \mathbb{R}^4 \) everywhere tangent to \( \mathcal{D} \). If we write \( \gamma(s) = (x(s), y(s), z(s), w(s)), s \in S^1 \), the condition for \( \gamma \) to be horizontal becomes

\[
\begin{align*}
z'(s) - y(s)x'(s) &= 0 \quad \text{and} \quad w'(s) - z(s)x'(s) = 0 \quad \text{for all} \quad s \in S^1.
\end{align*}
\]
A horizontal isotopy is an isotopy via horizontal loops. The rotation number \( \rho(\gamma) \) of a horizontal loop \( \gamma \) is the number of complete turns of the velocity vector \( \gamma'(s) \in D_{\gamma(s)} \), as we once traverse the loop in positive direction, relative to the trivialisation of \( D \) given by \( e_1, e_2 \). The rotation number is clearly invariant under horizontal isotopies, and it is easy to show by examples (see Sect. 3) that every integer can be realised as the rotation number of a horizontal loop.

The following theorem was proved by Adachi in [1]:

**Theorem** Two horizontal loops are horizontally isotopic if and only if their rotation numbers agree.

Adachi proves this theorem by studying the image of \( \gamma \) under the projection \( (x, y, z, w) \mapsto (x, w) \). (Beware that I have interchanged \( y \) and \( w \) in the definition of \( D \) compared with Adachi’s notation. This is more in line with the usual conventions as regards the contact geometric aspects of our discussion.) He determines the ‘Reidemeister moves’ in this projection and then reduces the proof to the corresponding classification of topologically trivial Legendrian knots in standard contact 3-space, due to Eliashberg and Fraser [2].

The purpose of the present note is to show that a much shorter proof can be given by using the projection \( (x, y, z, w) \mapsto (x, z) \) instead. See Sect. 4 for more on the relative merits of these projections.

### 2 Horizontal loops, Legendrian immersions and fronts

Let \( \gamma(s) = (x(s), y(s), z(s), w(s)), s \in S^1 \), be a horizontal loop. Notice that \( x'(s) = 0 \) implies \( z'(s) = w'(s) = 0 \) and hence—\( \gamma \) being an embedding—\( y'(s) \neq 0 \). This means that

\[
\overline{\gamma}(s) := (x(s), y(s), z(s)), \quad s \in S^1,
\]

defines a Legendrian immersion into \( \mathbb{R}^3 \) with its standard contact structure \( \xi := \ker(dz - ydx) \).

The rotation number \( \rho(\overline{\gamma}) \)—in the contact geometric sense—of such a Legendrian immersion is defined as the number of complete turns made by \( \overline{\gamma}'(s) \in \xi_{\overline{\gamma}(s)} \) as the loop \( \overline{\gamma} \) is traversed once in positive direction, relative to the trivialisation of \( \xi \) given by the vector fields

\[
\overline{e}_1 := \partial_x + y\partial_z \quad \text{and} \quad \overline{e}_2 := \partial_y.
\]

So it is obvious that \( \rho(\gamma) = \rho(\overline{\gamma}) \).

The rotation number \( \rho(\overline{\gamma}) \) is invariant under Legendrian regular homotopies, i.e. \( C^1 \)-homotopies via Legendrian immersions. Moreover, one can prove by elementary methods that the map \( [\overline{\gamma}] \mapsto \rho(\overline{\gamma}) \) defines a one-to-one correspondence between Legendrian regular homotopy classes of Legendrian immersions \( \overline{\gamma} : S^1 \to (\mathbb{R}^3, \xi) \) on the one hand, and the integers on the other; see [3, Theorem 6.3.10] or [4].