On some smooth projective two-orbit varieties with Picard number 1

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Abstract We classify all smooth projective horospherical varieties with Picard number 1. We prove that the automorphism group of any such variety $X$ acts with at most two orbits and that this group still acts with only two orbits on $X$ blown up at the closed orbit. We characterize all smooth projective two-orbit varieties with Picard number 1 that satisfy this latter property.

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0 Introduction

Horospherical varieties are complex normal algebraic varieties where a connected complex reductive algebraic group acts with an open orbit isomorphic to a torus bundle over a flag variety. The dimension of the torus is called the rank of the variety. Toric varieties and flag varieties are the first examples of horospherical varieties (see [14] for more examples and background).

It is well known that the only smooth projective toric varieties with Picard number 1 are the projective spaces. This is not the case for horospherical varieties: for example any flag variety $G/P$ with $P$ a maximal parabolic subgroup of $G$ is smooth, projective and horospherical with Picard number 1.

Moreover, smooth projective horospherical varieties with Picard number 1 are not necessarily homogeneous. For example, let $\omega$ be a skew-form of maximal rank on $\mathbb{C}^{2m+1}$. For $i \in \{1, \ldots, m\}$, define the odd symplectic grassmannian $\text{Gr}_\omega(i, 2m+1)$ as the variety of $i$-dimensional $\omega$-isotropic subspaces of $\mathbb{C}^{2m+1}$. Odd symplectic grassmannians are horospherical varieties (see Proposition 1.12) and, for $i \neq m$ they have...
two orbits under the action of their automorphism group which is a connected non-reductive linear algebraic group (see [13] for more details).

Our focus on smooth horospherical varieties with Picard number 1, that gives interesting examples of Fano horospherical varieties with Picard number 1, is also motivated by the main result of [14], where Fano horospherical varieties are classified in terms of rational polytopes. Indeed in [14, Th.0.1], the degree (i.e. the self-intersection number of the anticanonical bundle) of smooth Fano horospherical varieties is bounded. Two different bounds are obtained in the case of Picard number 1 and in the case of higher Picard number.

In Sect. 1, we classify all smooth projective horospherical varieties with Picard number 1. More precisely, we prove the following result.

**Theorem 0.1** Let $G$ be a connected reductive algebraic group. Let $X$ be a smooth projective horospherical $G$-variety with Picard number 1.

Then we have the following alternative:

(i) $X$ is homogeneous, or
(ii) $X$ is horospherical of rank 1. Its automorphism group is a connected non-reductive linear algebraic group, acting with exactly two orbits.

Moreover in the second case, $X$ is uniquely determined by its two closed $G$-orbits $Y$ and $Z$, isomorphic to $G/P_Y$ and $G/P_Z$, respectively; and $(G, P_Y, P_Z)$ is one of the triples of the following list.

1. $(B_m, P(\omega_{m-1}), P(\omega_m))$ with $m \geq 3$
2. $(B_3, P(\omega_1), P(\omega_3))$
3. $(C_m, P(\omega_i), P(\omega_{i+1}))$ with $m \geq 2$ and $i \in \{1, \ldots, m-1\}$
4. $(F_4, P(\omega_2), P(\omega_3))$
5. $(G_2, P(\omega_2), P(\omega_1))$

Here we denote by $P(\omega_i)$ the maximal parabolic subgroup of $G$ corresponding to the dominant weight $\omega_i$ with the notation of Bourbaki [4].

Remark that Case 3 of Theorem 0.1 corresponds to odd symplectic grassmannians. It would be natural to investigate other complete smooth spherical varieties with Picard number 1 (a normal variety is spherical if it admits a dense orbit of a Borel subgroup, for example horospherical varieties and symmetric varieties are spherical). A classification has been recently given in the special case of projective symmetric varieties by Ruzzi [16].

In the second part of this paper, we focus on another special feature of the non-homogeneous varieties classified by Theorem 0.1: the fact that they have two orbits even when they are blown up at their closed orbit. Two-orbit varieties (i.e. normal varieties where a linear algebraic group acts with two orbits) have already been studied by Akhiezer and Cupit-Foutou. In [1], Akhiezer classified those whose closed orbit is of codimension 1 and proved in particular that they are horospherical when the group is not semi-simple. In [8], Cupit-Foutou classified two-orbit varieties when the group is semi-simple, and she also proved that they are spherical. In Sect. 2, we define two smooth projective two-orbit varieties $X_1$ and $X_2$ with Picard number one (see Definitions 2.11 and 2.12) and we prove the following: