Cuspidal class number of the tower of modular curves $X_1(Np^n)$

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Abstract We consider a cuspidal class number, which is the order of a subgroup of the full cuspidal divisor class group of $X_1(Np^n)$ with $p \nmid N$ and $n \geq 1$. By studying the second generalized Bernoulli numbers, we obtain results similar to ones (Ferrero and Washington in Ann Math (2) 109(2):377–395, 1979; Washington in Invent Math 49:87–97, 1978) about the relative class numbers of cyclotomic $\mathbb{Z}_p$-extension of an abelian number field.

1 Introduction

Let $G$ be a finite abelian group with a surjective homomorphism $r : G \to (\mathbb{Z}/N\mathbb{Z})^\times$ for an integer $N > 0$. Let $\chi$ be a character on $G$. A generalized $k$th Bernoulli number $B_{k,\chi,G}$ can be defined for $\chi$ such that

$$B_{k,\chi,G} = N^{k-1} \sum_{g \in G} \chi(g) B_k \left( \frac{r(g)}{N} \right),$$

where $B_k(x)$ is the Bernoulli polynomial defined by the formula

$$B_k(x) = \sum_{r=0}^{k} \binom{k}{r} B_r x^{k-r}.$$
For \( G = (\mathbb{Z}/N\mathbb{Z})^\times \) and \( \chi \) a Dirichlet character modulo \( N \), \( B_{k,\chi,G} \) is the usual generalized Bernoulli numbers \( B_{k,\chi} \). In many different contexts, those generalized Bernoulli numbers have been related to an index of the Stickelberger ideal of order \( k \) in the group ring \( \mathbb{Z}[G] \), which is generated by a Stickelberger element

\[
\theta = N^{k-1} \sum_{g \in G} B_k \left( \frac{r(g)}{N} \right) g \in \mathbb{Q}[G]
\]

or by its variation.

When \( G = (\mathbb{Z}/N\mathbb{Z})^\times \) and \( k = 1 \), the relative class number \( h^-_N \) of \( \mathbb{Q}(\zeta_N) \) can be written as a product of \( B_{1,\chi} \) for odd Dirichlet characters \( \chi \). More precisely, one has

\[
h^-_N = Q_N w_N \prod_{\chi: \text{odd}} -\frac{1}{2} B_{1,\chi}, \tag{1.1}
\]

where \( Q_N \) is the unit index and \( w_N \) is the number of roots of unity. The relative class number \( h^-_N \) turns out to be the index of the minus part of the Stickelberger ideal of order 1. To the cyclotomic fields \( \mathbb{Q}(\zeta_{Np^n}) \), two non-negative integers \( \mu \) and \( \lambda \) are associated in order to express the \( p \)-adic valuation \( v_p(h^-_{Np^n}) \) of the relative class numbers \( h^-_{Np^n} \). In fact, by a theorem of Iwasawa one has

\[
v_p(h^-_{Np^n}) = \mu p^n + \lambda n + \nu \quad \text{for all } n \gg 1 \text{ and some } \nu \in \mathbb{Z}. \tag{1.2}
\]

It was conjectured by Iwasawa that \( \mu = 0 \) and it was proved by Ferrero and Washington [1].

Let \( \ell \) be an odd prime number different from \( p \). The \( \ell \)-adic valuation \( v_\ell(h^-_{Np^n}) \) of the relative class numbers was also determined by Washington [9]. He has shown that there exists a constant \( \delta_\ell \) such that

\[
v_\ell(h^-_{Np^n}) = \delta_\ell \quad \text{for all } n \gg 1 \tag{1.3}
\]

by verifying

**Theorem 1** [9] Let \( N \) be fixed and \( p \nmid N \). For almost all odd primitive Dirichlet characters \( \chi \) of modulo \( Np^n \), we have

\[
v_\ell(B_{1,\chi}) = 0.
\]

From the relative class number formula (1.1), one can conclude the formula (1.3).

In this paper, we consider the case that \( G = (\mathbb{Z}/N\mathbb{Z})^\times \) and \( k = 2 \). This case is studied by Kubert and Lang [4] for an odd prime power \( N = p^n \) and by Yu [12] for a general integer \( N > 4 \).

For an integer \( N > 4 \), a cusp on the modular curve \( X_1(N) \) is said to be of the first type if it is projected down to the cusp 0 on the modular curve \( X_0(q) \) for all prime divisor \( q \) of \( N \). We consider the group \( \overline{S}_1(N) \) of functions on \( X_1(N) \) whose divisors