Aubin’s Lemma for the Yamabe constants of infinite coverings and a positive mass theorem

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Abstract Aubin’s Lemma says that, if the Yamabe constant of a closed conformal manifold $(M, C)$ is positive, then it is strictly less than the Yamabe constant of any of its non-trivial finite conformal coverings. We generalize this lemma to the one for the Yamabe constant of any $(M_\infty, C_\infty)$ of its infinite conformal coverings, provided that $\pi_1(M)$ has a descending chain of finite index subgroups tending to $\pi_1(M_\infty)$. Moreover, if the covering $M_\infty$ is normal, the limit of the Yamabe constants of the finite conformal coverings (associated to the descending chain) is equal to that of $(M_\infty, C_\infty)$. For the proof of this, we also establish a version of positive mass theorem for a specific class of asymptotically flat manifolds with singularities.

1 Introduction and main results

There is a natural differential-topological invariant, called the Yamabe invariant, which arises from a variational problem for the functional $E$ below on a given closed smooth $n$-manifold $M$ with $n \geq 3$. It is well known that a Riemannian metric on $M$ is Einstein if and only if it is a critical point of the normalized Einstein-Hilbert functional $E$ on the space $\mathcal{M}(M)$ of all Riemannian metrics on $M$

$$E : \mathcal{M}(M) \to \mathbb{R}, \quad g \mapsto E(g) := \frac{\int_M R_g d\mu_g}{\text{Vol}_g(M)^{(n-2)/n}}.$$
Here, $R_g$, $d\mu_g$ and $\text{Vol}_g(M)$ denote respectively the scalar curvature, the volume element of $g$ and the volume of $(M, g)$.

Because the restriction of $E$ to any conformal class

$$[g] := \{ e^{2f} \cdot g \mid f \in C^\infty(M) \}$$

is bounded from below, we can consider the following conformal invariant (called the Yamabe constant of $[g]$)

$$Y(M, [g]) := \inf \{ E(\tilde{g}) \mid \tilde{g} = u^{4/(n-2)} \cdot g, u \in C^\infty_+(M) \}$$

$$= \inf \left\{ Q_{(M, g)}(u) := \frac{\int_M (\alpha_n |\nabla u|^2 + R_g u^2) d\mu_g}{\left( \int_M |u|^{2n/(n-2)} d\mu_g \right)^{(n-2)/n}} \mid u \in C^\infty_+(M), u \neq 0 \right\},$$

where $\alpha_n := \frac{4(n-1)}{n-2} > 0$ and

$$C^\infty_+(M) := \{ u \in C^\infty(M) \mid u > 0 \}.$$

A remarkable theorem [9,31,36,40,42] (cf. [10,29,37]) of Yamabe, Trudinger, Aubin, and Schoen asserts that each conformal class $[g]$ contains metrics $\tilde{g}$, called Yamabe metrics, for which

$$Y(M, [g]) = E(\tilde{g}).$$

A first variation argument shows that these metrics $\tilde{g}$ must have constant scalar curvature

$$R_{\tilde{g}} \equiv Y(M, [g]) \cdot \text{Vol}_{\tilde{g}}(M)^{-2/n}.$$

Hence, we call a conformal class $[g]$ positive if $Y(M, [g]) > 0$.

Let $C(M)$ denote the space of all conformal classes on $M$. The study of the second variation of $E$ done in [27,33] (cf. [12]) leads naturally to the definition of the following differential-topological invariant

$$Y(M) := \sup_{C \in C(M)} \inf_{g \in C} E(g).$$

This invariant is called the Yamabe invariant of $M$ and it was introduced independently by Kobayashi [24] and Schoen [32] (see also [25,33]). In other words, the Yamabe invariant $Y(M)$ of $M$ is the supremum of the scalar curvatures of unit-volume Yamabe metrics on $M$. We remark also that, for any $M$ and $C \in \mathcal{C}(M)$, Aubin [9] (cf. [10,29]) proved the following fundamental inequality

$$Y(M, C) \leq Y(S^n, [g_S]) = n(n-1)\text{Vol}_{g_S}(S^n)^{2/n}, \quad (1)$$