Leray’s inequality in general multi-connected
domains in $\mathbb{R}^n$

Reinhard Farwig · Hideo Kozono ·
Taku Yanagisawa

Received: 26 March 2010 / Revised: 5 August 2010 / Published online: 13 October 2011
© Springer-Verlag 2011

Abstract Consider the stationary Navier–Stokes equations in a bounded domain
$\Omega \subset \mathbb{R}^n$ whose boundary $\partial \Omega$ consists of $L + 1$ smooth $(n - 1)$-dimensional closed
hypersurfaces $\Gamma_0, \Gamma_1, \ldots, \Gamma_L$, where $\Gamma_1, \ldots, \Gamma_L$ lie inside of $\Gamma_0$ and outside of one
another. The Leray inequality of the given boundary data $\beta$ on $\partial \Omega$ plays an important
role for the existence of solutions. It is known that if the flux $\gamma_i \equiv \int_{\Gamma_i} \beta \cdot \nu dS = 0$
on $\Gamma_i (\nu$: the unit outer normal to $\Gamma_i$) is zero for each $i = 0, 1, \ldots, L$, then the Leray
inequality holds. We prove that if there exists a sphere $S$ in $\Omega$ separating $\partial \Omega$ in such
a way that $\Gamma_1, \ldots, \Gamma_k$ $(1 \leq k \leq L)$ are contained inside of $S$ and that the others
$\Gamma_{k+1}, \ldots, \Gamma_L$ are outside of $S$, then the Leray inequality necessarily implies that
$\gamma_1 + \cdots + \gamma_k = 0$. In particular, suppose that there are $L$ spheres $S_1, \ldots, S_L$ in $\Omega$
lying outside of one another such that $\Gamma_i$ lies inside of $S_i$ for all $i = 1, \ldots, L$. Then
the Leray inequality holds if and only if $\gamma_0 = \gamma_1 = \cdots = \gamma_L = 0$. 

Dedicated to Professor Izumi Takagi on the occasion of his 60th birthday.

R. Farwig
Department of Mathematics, Darmstadt University of Technology,
64289 Darmstadt, Germany
e-mail: farwig@mathematik.tu-darmstadt.de

H. Kozono (✉)
Mathematical Institute, Tohoku University, Sendai 980-8578, Japan
e-mail: kozono@math.tohoku.ac.jp

T. Yanagisawa
Department of Mathematics, Nara Women’s University, Nara 630-8506, Japan
e-mail: taku@cc.nara-wu.ac.jp
1 Introduction

We consider Leray’s problem on the stationary Navier–Stokes equations with inhomogeneous boundary data under the general flux condition. Let \( \Omega \subset \mathbb{R}^n, n \geq 2, \) be a bounded domain with smooth boundary \( \partial \Omega. \) Throughout this paper, we impose the following assumption on \( \Omega. \)

**Assumption** The boundary \( \partial \Omega \) has \( L + 1 \) connected components \( \Gamma_0, \Gamma_1, \ldots, \Gamma_L \) of \( (n - 1) \)-dimensional \( C^\infty \)-closed hypersurfaces such that \( \Gamma_1, \ldots, \Gamma_L \) lie inside of \( \Gamma_0 \) and outside of one other;

\[
\partial \Omega = \bigcup_{j=0}^L \Gamma_j.
\]

In the most interesting case when \( n = 3, \) it is often called that \( \Omega \) has the second Betti number \( L. \) In \( \Omega \) we consider the boundary value problem for the stationary Navier–Stokes equations:

\[
\begin{cases}
-\mu \Delta v + v \cdot \nabla v + \nabla p = 0 & \text{in } \Omega, \\
div v = 0 & \text{in } \Omega, \\
v = \beta & \text{on } \partial \Omega,
\end{cases}
\]

(N-S)

where \( v = v(x) = (v_1(x), \ldots, v_n(x)) \) and \( p = p(x) \) denote the unknown velocity vector and the unknown pressure at the point \( x = (x_1, \ldots, x_n) \in \Omega, \) while \( \mu > 0 \) is the given viscosity constant, and \( \beta = \beta(x) = (\beta_1(x), \ldots, \beta_n(x)) \) is the given boundary data on \( \partial \Omega. \) We use the standard notation as \( \Delta v = \sum_{j=1}^n \frac{\partial^2 v}{\partial x_j^2}, \nabla p = \left( \frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n} \right), \) \( \text{div } v = \sum_{j=1}^n \frac{\partial v_j}{\partial x_j}, \) and \( v \cdot \nabla v = \sum_{j=1}^n v_j \frac{\partial v}{\partial x_j}. \) Since the solution \( v \) satisfies \( \text{div } v = 0 \) in \( \Omega, \) the given boundary data \( \beta \) on \( \partial \Omega \) is required to fulfill the following compatibility condition which we call the general flux condition:

\[
\sum_{j=0}^L \int_{\Gamma_j} \beta \cdot v dS = 0,
\]

(G.F.)

where \( v \) denotes the unit outer normal to \( \partial \Omega. \) Leray [10] proposed to solve the following problem.

**Leray’s problem.** Let \( n = 2, 3. \) Suppose that \( \beta \in H^{1/2}(\partial \Omega) \) satisfies the general flux condition \( (G.F.). \) Does there exist at least one weak solution \( v \in H^1(\Omega) \) of (N-S)?

Up to now, we are not yet successful to give a complete answer to this question. However, some partial answer has been proved by Leray [10], Fujita [3] and Ladyzhenskaya [9] under the restricted flux condition \( (R.F.) \) on \( \beta: \)

\[
\gamma_j \equiv \int_{\Gamma_j} \beta \cdot v dS = 0 \quad \text{for all } j = 0, 1, \ldots, L.
\]

(R.F.)