Leray’s inequality in general multi-connected domains in $\mathbb{R}^n$

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Abstract Consider the stationary Navier–Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary $\partial \Omega$ consists of $L+1$ smooth $(n-1)$-dimensional closed hypersurfaces $\Gamma_0, \Gamma_1, \ldots, \Gamma_L$, where $\Gamma_1, \ldots, \Gamma_L$ lie inside of $\Gamma_0$ and outside of one another. The Leray inequality of the given boundary data $\beta$ on $\partial \Omega$ plays an important role for the existence of solutions. It is known that if the flux $\gamma_i \equiv \int_{\Gamma_i} \beta \cdot \nu dS = 0$ on $\Gamma_i (\nu$: the unit outer normal to $\Gamma_i)$ is zero for each $i = 0, 1, \ldots, L$, then the Leray inequality holds. We prove that if there exists a sphere $S$ in $\Omega$ separating $\partial \Omega$ in such a way that $\Gamma_1, \ldots, \Gamma_k$ ($1 \leq k \leq L$) are contained inside of $S$ and that the others $\Gamma_{k+1}, \ldots, \Gamma_L$ are outside of $S$, then the Leray inequality necessarily implies that $\gamma_1 + \cdots + \gamma_k = 0$. In particular, suppose that there are $L$ spheres $S_1, \ldots, S_L$ in $\Omega$ lying outside of one another such that $\Gamma_i$ lies inside of $S_i$ for all $i = 1, \ldots, L$. Then the Leray inequality holds if and only if $\gamma_0 = \gamma_1 = \cdots = \gamma_L = 0$.

Dedicated to Professor Izumi Takagi on the occasion of his 60th birthday.

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1 Introduction

We consider Leray’s problem on the stationary Navier–Stokes equations with inhomogeneous boundary data under the general flux condition. Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \), be a bounded domain with smooth boundary \( \partial \Omega \). Throughout this paper, we impose the following assumption on \( \Omega \).

**Assumption** The boundary \( \partial \Omega \) has \( L + 1 \) connected components \( \Gamma_0, \Gamma_1, \ldots, \Gamma_L \) of \( (n - 1) \)-dimensional \( C^\infty \)-closed hypersurfaces such that \( \Gamma_1, \ldots, \Gamma_L \) lie inside of \( \Gamma_0 \) and outside of one another;

\[
\partial \Omega = \bigcup_{j=0}^L \Gamma_j.
\]

In the most interesting case when \( n = 3 \), it is often called that \( \Omega \) has the second Betti number \( L \). In \( \Omega \) we consider the boundary value problem for the stationary Navier–Stokes equations:

\[
\begin{aligned}
-\mu \Delta v + v \cdot \nabla v + \nabla p &= 0 \quad \text{in } \Omega, \\
\text{div } v &= 0 \quad \text{in } \Omega, \\
v &= \beta \quad \text{on } \partial \Omega,
\end{aligned}
\]

(N-S)

where \( v = v(x) = (v_1(x), \ldots, v_n(x)) \) and \( p = p(x) \) denote the unknown velocity vector and the unknown pressure at the point \( x = (x_1, \ldots, x_n) \in \Omega \), while \( \mu > 0 \) is the given viscosity constant, and \( \beta = \beta(x) = (\beta_1(x), \ldots, \beta_n(x)) \) is the given boundary data on \( \partial \Omega \). We use the standard notation as \( \Delta v = \sum_{j=1}^n \frac{\partial^2 v}{\partial x_j^2}, \nabla p = \left( \frac{\partial p}{\partial x_1}, \ldots, \frac{\partial p}{\partial x_n} \right) \), \( \text{div } v = \sum_{j=1}^n \frac{\partial v_j}{\partial x_j} \), and \( v \cdot \nabla v = \sum_{j=1}^n v_j \frac{\partial v}{\partial x_j} \). Since the solution \( v \) satisfies \( \text{div } v = 0 \) in \( \Omega \), the given boundary data \( \beta \) on \( \partial \Omega \) is required to fulfill the following compatibility condition which we call the general flux condition:

\[
\sum_{j=0}^L \int_{\Gamma_j} \beta \cdot v dS = 0,
\]

(G.F.)

where \( \nu \) denotes the unit outer normal to \( \partial \Omega \). Leray [10] proposed to solve the following problem.

**Leray’s problem.** Let \( n = 2, 3 \). Suppose that \( \beta \in H^{1/2}(\partial \Omega) \) satisfies the general flux condition (G.F.). Does there exist at least one weak solution \( v \in H^1(\Omega) \) of (N-S)?

Up to now, we are not yet successful to give a complete answer to this question. However, some partial answer has been proved by Leray [10], Fujita [3] and Ladyzhenskaya [9] under the restricted flux condition (R.F.) on \( \beta \):

\[
\gamma_j \equiv \int_{\Gamma_j} \beta \cdot v dS = 0 \quad \text{for all } j = 0, 1, \ldots, L.
\]

(R.F.)