Fuchsian groups, automorphic functions and Schwarzians

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1. Introduction

Let $G$ be a Fuchsian group of the first kind acting on the upper half-plane $\mathbb{H}$ such that the compactification $X$ of the open Riemann surface $G\backslash \mathbb{H}$ has genus zero; we then say that $G$ is of genus zero. If a function $f$ defined on $X$ generates the function field over $\mathbb{C}$ of $X$, then $f$ is called a Hauptmodul for the genus zero group $G$, and is defined up to linear fractional transformations. Each Hauptmodul can be extended to a meromorphic function defined on $\mathbb{H}$ to become an automorphic function with respect to $G$. A prototype is the elliptic modular function $j$ which is a Hauptmodul for the modular group $\text{PSL}_2(\mathbb{Z})$. When $G$ contains the transformation $\tau \to \tau + 1$ which generates its translation subgroup (the stabilizer of $\infty$), then each Hauptmodul has a Fourier expansion in $q = \exp(2\pi i \tau)$, and one of them has the form

$$f(\tau) = \frac{1}{q} + \sum_{n \geq 1} a_n q^n, \quad a_n \in \mathbb{C}.$$ (1.1)

To such an $f$ and for each positive integer $n$, there exists a unique monic polynomial of degree $n$, $P_n = P_n(f)$ whose coefficients depend on the coefficients $\{a_k\}$ of $f$. It is characterized by the property that $P_n(f) - 1/q^n$ is a power series in $q$. 

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with no constant term, see Sect. 4. For example, if \( f \) is the elliptic modular function \( j \) normalized to have the form (1.1), then

\[
P_n(j) = nT_n(j),
\]

where \( T_n \) is the classical \( n \)-th Hecke operator.

For a meromorphic function \( f \) defined on some complex domain, there is a differential operator known as the Schwarzian (or the Schwarz derivative) defined by

\[
\{ f, z \} = 2 \left( \frac{f''}{f'} \right)' - \left( \frac{f''}{f'} \right)^2.
\]

which is invariant under linear fractional transformations of \( f \). For \( f \) given formally by (1.1), the Schwarzian of \( f \) is completely described in terms of the \( q \)-coefficients of the Faber polynomial \( P_n(f) \), see Proposition 4.1.

For an automorphic function \( f \) and for a Fuchsian group \( G \), we find \( \{ f, \tau \} \) to be an automorphic form of weight 4 for \( G \). When \( f \) is a Hauptmodul, then \( \{ f, \tau \} \) is generically invariant under a larger group, namely the normalizer of \( G \) in \( \text{PSL}_2(\mathbb{R}) \), and for the inverse function \( \tau(f) \), \( \{ \tau, f \} \) is a rational function of \( f \). The analytic behaviour of \( \{ f, \tau \} \) is that it is holomorphic in \( \mathfrak{H} \) except at elliptic fixed points where it has poles of order 2, and it is holomorphic at the cusps, see Proposition 6.2.

This leads us to restrict our attention to genus zero Fuchsian groups with no elliptic elements. For finiteness reasons [11] we consider only those groups which contain some \( \Gamma_0(n) \) with finite index and such that the stabilizer of \( \infty \) is generated by \( \tau \to \tau + 1 \). In other words, these are torsion-free genus zero groups with the cusp at \( \infty \) having width 1. We determine all the \( n \) such that \( \Gamma_0(n) \), or a conjugate, satisfies these conditions. There are 14 such groups which have Hauptmoduls given by eta-products, and there are only 3 more groups which are not \( \Gamma_0(n) \) or a conjugate with the same property.

The Schwarzian of a Hauptmodul for such a group, being holomorphic on \( \mathfrak{H} \) and at the cusps, is completely determined in terms of a canonical weight 4 automorphic form. For 14 groups, these forms are theta functions of variously normed rank 8 lattices (Sect. 7), and for the 3 remaining cases they are simple linear combinations of Eisenstein series and known cusp forms (Sect. 8). The theta functions arise only when the groups are, up to conjugacy, \( \Gamma_0(n) \). The significance of the lattices of the theta functions involved is as yet unknown.