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Dold-Kan type theorem for Γ-groups

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0. Introduction

By a well-known theorem of Dold-Kan the Moore normalization establishes an equivalence between the category of simplicial abelian groups and the category of chain complexes (see [6]). Our main result shows that there is a similar theorem for Γ-groups. Here Γ is the category of finite based sets. For a functor $F : \Gamma \to Groups$, we construct a functor $cr(F) : \Omega \to Groups$, where Ω is the category of all nonempty finite sets and surjections. This construction is based on the notion of cross-effects of functors [4], which is a generalization of the classical definition of Eilenberg and Mac Lane [8] to the non-abelian setup. Our version of the Dold-Kan theorem in the abelian case claims that the category of abelian Γ-groups is equivalent to the category of functors Ab^{Ω} and the equivalence is given by $F \mapsto cr(F)$. In the non-abelian case one needs to consider functors $T : \Omega \to Gr$ together with maps

$$\{,\} : T_n \times T_m \to T_{n+m}$$

satisfying additional relations. Here T_n is the value of T on the set $\underline{n} := \{1, \cdots, n\}$. A Γ-group F is called polynomial if $cr(F)_n = 0$ for $n >> 0$. It is a consequence of our Dold-Kan type theorem for Γ-groups, that up to obvious induction one can reduce the study of polynimial Γ-groups to the abelian case. As a sample application of this principle we prove that for any finite simplicial set K and for any polynomial Γ-group F one has $\pi_n(F(K)) = 0$ for all $n > (dim\ K)(deg\ F)$.

Our interest in Γ-groups comes from the famous result of Segal [16], who proved that Γ-spaces are combinatorial models for connective spectra (see also [1], [5]). Based on the Kan-Thurston theorem we show that any connective spectrum can be obtained from a discrete Γ-group up to stable weak equivalence. Our next goal is to get information about homotopy of spectra corresponding to

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polynomial Γ-groups. If a Γ-group is of degree 1, then the corresponding spectrum is an Eilenberg-MacLane spectrum. However in general spectra associated to polynomial Γ-groups are not products of suspensions of Eilenberg -MacLane spectra even in the degree 2 case (see [3]). Of course the spectra associated to abelian Γ-groups are products of suspensions of Eilenberg -MacLane spectra. However to make explicit calculations is a very hard problem. We construct a spectral sequence, whose abutment is the stable homotopy of abelian Γ-groups and which gives a nice relationship between the homology of the symmetric groups and the stable homotopy of Γ-groups. Our spectral sequence is in the same spirit as the spectral sequence of Lück [11] (see also [17]), and uses in an essential way the fact that the category Ω is an so called EI-category (see [11]). As an application of our spectral sequence we give the calculation of the stable homotopy of any Γ-group of degree ≤ 3. We give also a calculation of the l-torsion part of the stable homotopy of any Γ-group of degree $\leq l$ in demensions $\geq l$. Here l is any prime. In the last section we give a complete calculation of the l-torsion part of the Dold-Puppe stable derived functors of functors of degree $\leq l$.

1. Γ-spaces and Γ-groups

1.1. Stable homotopy of Γ-spaces. Let Γ be the small category of finite pointed sets. We will assume that objects of Γ are sets $[n]$, where

$$[n] := \{0, 1, ..., n\}$$

with basepoint 0, $n \geq 0$. Let $Sets_*$ be the category of all pointed sets. Let $\mathcal{C}$ be a category with zero object. A *Γ-object over $\mathcal{C}$* is a functor $A : \Gamma \to \mathcal{C}$ such that $A([0]) = 0$. Let $\mathcal{C}^\Gamma$ be the category of Γ-objects over $\mathcal{C}$. By definition a *pointed Γ-set* is the same as a Γ-object over the category of pointed sets. Similarly an *abelian Γ-group* or a *Γ-group* is the same as a Γ-object over the category of abelian groups or groups respectively. A *Γ-space* is the same as a Γ-object over the category $s.Sets_*$ of pointed simplicial sets. Hence any Γ-set (and therefore any Γ-group) yields a discrete Γ-space. A natural transformation $F \to T$ of Γ-spaces is called a *pointwise fibration* (resp. *pointwise weak equivalence*) if $F([n]) \to T([n])$ is a fibration (resp. weak equivalence) for any $n \geq 0$. Any Γ-space F gives rise to a binatural transformation

$$X \wedge F(Y) \to F(X \wedge Y)$$

as follows. For any $x \in X$ define $\hat{x} : Y \to X \wedge Y$, by $y \mapsto (x, y)$. Then apply F to get $F(\hat{x}) : F(Y) \to F(X \wedge Y)$. Now one defines $X \wedge F(Y) \to F(X \wedge Y)$, by $(x, z) \mapsto F(\hat{x})(z)$. There is a standard way to prolong a Γ-space F to a functor $s.Sets_* \to s.Sets_*$. First using direct limits one can prolong F to a