An obstruction to the existence of Einstein metrics on 4-manifolds

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1 Introduction

There exist two well known topological obstructions to the existence of Einstein metrics on a differentiable compact oriented 4-manifold $Y$.

The first one is Thorpe’s inequality, that comes from the Gauss-Bonnet-Chern formula for the Euler characteristic $\chi(Y)$ of $Y$ and from the Hirzebruch formula for the signature $\tau(Y)$ of $Y$ (see [3]), which allow us to express these two topological invariants in terms of the irreducible components of the curvature under the action of $SO(4)$. It can be stated in the following way.

**Theorem 1.1 (J. Thorpe-N. Hitchin, [3], p.210-[13])** Let $Y$ be a compact oriented manifold of dimension 4. If $\chi(Y) < \frac{1}{2} |\tau(Y)|$ then $Y$ doesn’t admit any Einstein metric.

Moreover, if $\chi(Y) = \frac{1}{2} |\tau(Y)|$ then $Y$ admits no Einstein metric unless it is either flat or a K3 surface or an Enriques surface or the quotient of an Enriques surface by a free antiholomorphic involution.

This theorem implies a previous result of M. Berger who proved that there exists no compact Einstein 4-manifold with negative Euler characteristic.

On the other hand, combining the Gauss-Bonnet-Chern formula for the Euler characteristic with Gromov’s estimation of simplicial volume $\| Y \|$ of a riemannian manifold $Y$ (see [9], p.12), M. Gromov obtained the following obstruction:

**Theorem 1.2 (M. Gromov [9])** Let $Y$ be a compact manifold of dimension 4. If $\chi(Y) < \frac{1}{2592\pi^2} \| Y \|$ then $Y$ doesn’t admit any Einstein metric.

As Gromov already remarked, this inequality is not sharp (we shall make it evident on some examples); in fact, we shall give an improvement of this result
which, in some cases, gives a sharp inequality. Moreover, the simplicial volume is not so easy to compute explicitly as the Euler characteristic and the signature are.

We shall frequently refer to the inequalities of Theorems 1.1 and 1.2 respectively as to Thorpe’s and Gromov’s inequalities or obstruction conditions (watch out that, for historical reasons, most people mean by Thorpe’s and Gromov’s inequalities the trivially equivalent result which says that the inverse inequalities are satisfied by any Einstein manifold).

The above formulas enable to exhibit some examples of differentiable 4-manifolds (even simply connected, like \((\mathbb{C}P^2)^n\) for \(n \geq 4\)) with arbitrarily high Euler characteristic which don’t admit any Einstein metric.

The purpose of this paper is to show that, under the topological hypothesis that \(Y\) admits a non-zero degree map on some compact locally symmetric 4-manifold (that is, a real or complex hyperbolic 4-manifold), additional obstructions arise on \(\chi(Y)\) and \(\tau(Y)\). Some of these obstructions are sharp inequalities, that is the equality case contains one Einstein manifold and only one: the locally symmetric one.

We stress the fact that by “topological obstruction to the existence of Einstein metrics” we mean an obstruction (in terms of topological invariants) to the existence of any Einstein metric: that is, not only of Einstein metrics of a specified sign or Kähler-Einstein metrics, for which other results are known (see respectively [2], [5]).

Our new obstructions produce a lot of 4-manifolds which admit no Einstein metric and, nevertheless, do not satisfy Thorpe’s and Gromov’s inequalities (Sect. 3).

In Sect. 4 we shall show a property of genericity of non-Einstein 4-manifolds, namely that for any compact 4-manifold \(Y\) there exist infinitely many (non-homeomorphic) compact 4-manifolds \(Y_i\), which have the same signature and the same Euler characteristic of \(Y\), and which admit no Einstein metric.

Section 5 provides examples of complex surfaces without Einstein metrics, which don’t satisfy the classical obstructions.

Throughout the paper, every manifold is supposed to be differentiable, 4-dimensional, compact and connected (unless otherwise stated).

2 The general case

A riemannian manifold \((X, g_0)\) will be called real hyperbolic (resp. complex hyperbolic) when it is a quotient of the real hyperbolic space form, i.e. its curvature is constant and equal to -1 (resp. when \((X, g_0)\) is a quotient of the complex hyperbolic space, with curvature normalized between -4 and -1).

The volume entropy of a riemannian manifold \((Y, g)\) is defined to be

\[
h(g) = \lim_{r \to +\infty} \frac{1}{r} \ln \text{Vol}(\hat{B}(x, r))
\]