On Finsler geometry of submanifolds

Zhongmin Shen

Department of Mathematics, IUPUI, 402 N. Blackford Street, Indianapolis, IN 46202-3216 USA
(e-mail: zshen@math.iupui.edu)

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1. Introduction

The differential geometry of Riemann submanifolds has been well developed by many geometers. There are many important local and global results, which in turn lead to a better understanding on Riemann manifolds [C1][C2][C5][D][W2]. In the general case, however, Finsler submanifolds have not been studied at the same pace. Several people have made some fundamental contributions to this subject from various points of view [AZ] [Ba1]–[Ba3] [Be1], [Be2] [Co1]–[Co3] [Da] [Dr] [Ho] [M1]–[M3] [R1]–[R3] [Sa] [We1][We2], etc. The most natural idea is to study the induced (and intrinsic) connections and establish some equations which relate the curvatures of submanifolds and the curvatures of the ambient manifold.

The purpose of this paper is to introduce the notions of mean curvature and normal curvature, then establish some global results for submanifolds in a Minkowski space. We shall avoid using any connection so that the reader can understand the arguments here without the connection theory in Finsler geometry.

Minkowski spaces are the simplest Finsler manifolds. Let $V^n$ denote the standard $n$-dimensional real vector space. A Minkowski space is $V^n$ equipped with a Minkowski norm $F$ (whose indicatrix is strongly convex). If $F$ is Euclidean, then denote $\mathbb{R}^n = (V^n, F)$. Roughly speaking, a Finsler metric $F$ on an $n$-manifold $M$ is a collection of Minkowski norms $F_x$ in $T_x M$ such that $F_x$ varies smoothly in $x$. $F$ is called Riemannian if $F_x$ is Euclidean for all $x \in M$. Apparently, Finsler metrics are much more complicated than Riemann metrics. There are many problems from other areas related to Finsler metrics [AIM].

First we consider the isometric imbedding problem. Let $(M, F)$ be a closed Finsler manifold. $F$ induces an inner metric $d = d_F$ by integrals. Let $L^\infty$ denote
the completion of the norm space of continuous functions $f$ on $M$, whose norm is given by $\|f\|_\infty := \sup_{x \in M} |f(x)|$. Then $L^\infty$ is an infinite-dimensional Banach space. There is a natural imbedding $\Phi : M \to L^\infty$ given by $\Phi(x) = d_x$, where $d_x$ denotes the distance function from $x$. It is easy to prove that
\[
\|\Phi(x_0) - \Phi(x_1)\|_\infty = d(x_0, x_1), \quad \forall x_0, x_1 \in M.
\]
(Compare [G]). This means that every closed Finsler manifold can be isometrically embedded into a Banach space $L^\infty$. One of the fundamental problems in Finsler geometry is whether or not every Finsler manifold can be isometrically immersed into a Minkowski space (which is a finite-dimensional Banach space)?

The answer is affirmative for Riemann manifolds. In 1957, J. Nash [N] proved that any $n$-dimensional Riemannian manifold can be isometrically imbedded into an $m$-dimensional Euclidean space ($m = n(3n + 11)/2$ in the compact case, and $m = 2(2n + 1)(3n + 8)$ in the non-compact case). However for general Finsler manifolds, the problem becomes very difficult. R. Ingarden [I] proves that every $n$-dimensional Finsler manifold can be locally isometrically imbedded into a $2n$-dimensional “weak” Minkowski space (i.e., the indicatrix is not necessarily strongly convex). Later on, C. H. Gu [Gu1] asserts that every $n$-dimensional Finsler manifold can be locally isometrically imbedded into a $2n$-dimensional Minkowski space (in our sense). C. H. Gu also claims a global imbedding theorem [Gu2]. But it seems that he can only prove the existence of a global isometric imbedding into a “weak” Minkowski space. Recently, D. Burago and S. Ivanov [BI] successfully prove that any compact $C^r$ manifold ($r \geq 3$) with a $C^2$ Finsler metric admits a $C^r$ imbedding into a finite-dimensional Banach space ($\mathbb{V}^m, \|\cdot\|$).

The author believes that $\|\cdot\|$ can be a Minkowski norm. In [BI], Burago-Ivanov also assert that for any non-compact manifold $M$, there exists a Finsler metric $F$ such that $(M, F)$ cannot be imbedded into any finite-dimensional Banach space.

In this paper, we shall construct an explicit Finsler metric $F$ on $\mathbb{R}^2$ such that $\mathbb{R}^2, F)$ cannot be isometrically imbedded into any Minkowski space. The example is constructed based on the following

**Theorem 1.1** Suppose that a Finsler manifold $(M, F)$ can be isometrically imbedded into a Minkowski space $(\mathbb{V}^m, F)$. Then the Cartan tensor $A$ of $F$ and the Cartan tensor $\tilde{A}$ of $\tilde{F}$ satisfy $\sup_{x \in M} \|A\| \leq \|\tilde{A}\| < \infty$. Thus, Finsler manifolds with unbounded Cartan tensor cannot be isometrically imbedded into any Minkowski space.

Let $F : \mathbb{T}^2 \to [0, \infty)$ be given by
\[
F(Y) = \sqrt{|y_1|^2 + |y_2|^2 + \varphi(x)\sqrt{|y_1|^4 + |y_2|^4}}, \quad Y = y^i \frac{\partial}{\partial x^i} \in \mathbb{T}^2,
\]
where $\varphi : \mathbb{R}^2 \to [0, \infty)$ is an arbitrary nonnegative smooth function with $\sup \varphi = \infty$. We shall prove that $\|A\|$ is unbounded on $\mathbb{R}^2$. According to Theorem 1.1, $(\mathbb{R}^2, F)$ cannot be isometrically imbedded into any Minkowski space.