Hyponormality of Toeplitz operators
with polynomial symbols

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1 Introduction

A bounded linear operator $A$ on a Hilbert space $H$ with inner product $(\cdot, \cdot)$ is said to be hyponormal if its selfcommutator $[A^*, A] = A^*A - AA^*$ induces a positive semidefinite quadratic form on $H$ via $\xi \mapsto ([A^*, A] \xi, \xi)$, for $\xi \in H$. Let $H^2(T)$ denote the Hardy space of the unit circle $T = \partial \mathbb{D}$ in the complex plane. Recall that given $\varphi \in L^\infty(T)$, the Toeplitz operator with symbol $\varphi$ is the operator $T_\varphi$ on $H^2(T)$ defined by $T_\varphi f = P(\varphi \cdot f)$, where $f \in H^2(T)$ and $P$ denotes the projection that maps $L^2(T)$ onto $H^2(T)$. The hyponormality of Toeplitz operators has been studied by C. Cowen [1], [2], P. Fan [4], C. Gu [8], T. Ito and T. Wong [9], T. Nakazi and K. Takahashi [11], D. Yu [13], K. Zhu [14], R. Curto, D. Farenick, the second and the third named authors [3], [5], [6], [10] and others. An elegant theorem of C. Cowen [2] characterizes the hyponormality of a Toeplitz operator $T_\varphi$ on $H^2(T)$ by properties of the symbol $\varphi \in L^\infty(T)$. K. Zhu [14] reformulated Cowen’s criterion and then showed that the hyponormality of $T_\varphi$ with polynomial symbols $\varphi$ can be decided by a method based on the classical interpolation theorem of I. Schur [12]. Also Farenick and the third named author [5] characterized the hyponormality of $T_\varphi$ in terms of the Fourier coefficients of the trigonometric polynomial $\varphi$ in the cases that the outer coefficients of $\varphi$ have the same modulus. But the case of arbitrary trigonometric polynomials $\varphi$, though

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solved in principle by Cowen’s theorem or Zhu’s theorem, is in practice very complicated. On the other hand, Nakazi and Takahashi [11, Corollary 5] showed that if \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n \) is a trigonometric polynomial with \( m \leq N \) and if for every zero \( \zeta \) of \( z^m \varphi \) such that \( |\zeta| > 1 \), the number \( 1/\zeta \) is a zero of \( z^m \varphi \) in the open unit disk \( \mathbb{D} \) of multiplicity greater than or equal to the multiplicity of \( \zeta \), then \( T_{\varphi} \) is hyponormal. But the converse is not true in general. To see this consider the following trigonometric polynomial: \[ \varphi(z) = z^{-2}(z-2)(z-1)(z-\frac{1}{3}). \] Then \( \varphi(z) = \frac{2}{15} z^{-2} - \frac{19}{15} z^{-1} + \frac{55}{15} - \frac{53}{15} z + z^2 \). Using an argument of P. Fan [4, Theorem 1] – for every trigonometric polynomial \( \varphi \) of the form \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n \),

\[ (0.1) \quad \text{if } T_{\varphi} \text{ is hyponormal } \iff \left| \det \left( \frac{a_1}{\tau_1} \frac{a_2}{\tau_2} \right) \right| \leq |a_2|^2 - |a_2|^2, \]

a straightforward calculation shows that \( T_{\varphi} \) is hyponormal. In this paper we consider how the converse of the above result due to Nakazi and Takahashi survives for arbitrary trigonometric polynomials. The main results are as follows. Suppose \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n \) with \( m \leq N \) and write

\[ \mathfrak{F} := \{ \zeta, 1/\zeta : \text{the complex numbers } \zeta \text{ and } 1/\zeta \text{ are zeros of } z^m \varphi \}. \]

If \( \mathfrak{F} \) contains at least \((N + 1)\) elements then the following statements are equivalent.

(i) \( T_{\varphi} \) is a hyponormal operator.

(ii) For every zero \( \zeta \) of \( z^m \varphi \) such that \( |\zeta| > 1 \), the number \( 1/\zeta \) is a zero of \( z^m \varphi \) in the open unit disk \( \mathbb{D} \) of multiplicity greater than or equal to the multiplicity of \( \zeta \).

Moreover, in the cases where \( T_{\varphi} \) is a hyponormal operator, the rank of the selfcommutator of \( T_{\varphi} \) is computed from the formula \( \text{rank } [T_{\varphi}, T_{\varphi}] = N - m + Z_{\mathbb{D}} - Z_{\mathbb{C}\setminus\mathbb{D}} \), where \( Z_{\mathbb{D}} \) and \( Z_{\mathbb{C}\setminus\mathbb{D}} \) are the number of zeros of \( z^m \varphi \) in \( \mathbb{D} \) and in \( \mathbb{C}\setminus\mathbb{D} \) counting multiplicity. In addition, a new necessary condition for hyponormality of \( T_{\varphi} \) with polynomial symbols \( \varphi \) is presented: if \( \varphi(z) = \sum_{n=-m}^{N} a_n z^n \) is such that \( T_{\varphi} \) is hyponormal and if \( z^m \varphi = a_N \prod_{j=1}^{m+N} (z - \zeta_j) \), then

\[ \left| \sum_{j=1}^{m+N} (\zeta_j - 1/\zeta_j) \right| \leq \frac{1}{\prod_{j=1}^{m+N} |\zeta_j|} - \prod_{j=1}^{m+N} |\zeta_j|. \]

2 Main results

We shall use a variant of Cowen’s theorem [1] that was first proposed by Nakazi and Takahashi [11].

**Cowen’s Theorem.** Suppose \( \varphi \in L^\infty(\mathbb{T}) \) is arbitrary and write

\[ \mathcal{E}(\varphi) = \{ k \in H^\infty(\mathbb{T}) : ||k||_\infty \leq 1 \text{ and } \varphi - k\varphi \in H^\infty(\mathbb{T}) \}. \]