Seshadri constants on algebraic surfaces

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0 Introduction

Seshadri constants are local invariants that are naturally associated to polarized varieties. Except in the simplest cases they are very hard to control or to compute explicitly. The purpose of the present paper is to study these invariants on algebraic surfaces; we prove a number of explicit bounds on Seshadri constants and Seshadri sub-maximal curves, and we give complete results for abelian surfaces of Picard number one.

In recent years there has been considerable interest in understanding the local positivity of ample line bundles on algebraic varieties. Motivated in part by the study of linear series in connection with Fujita’s conjectures, Demailly
[12] captured the concept of local positivity in the *Seshadri constant*, a real number \( \varepsilon(L, x) \) associated with an ample line bundle \( L \) at a point \( x \) of an algebraic variety \( X \), which in effect measures how much of the positivity of \( L \) can be concentrated at \( x \). Interest in Seshadri constant derives on the one hand from the fact that, via vanishing theorems, a lower bound on the Seshadri constants \( \varepsilon(L, x) \) yields bounds on the number of points and jets that the adjoint series \( \mathcal{O}_X(K_X + L) \) separates. On the other hand it has become increasingly clear that Seshadri constants are highly interesting invariants of algebraic varieties quite in their own right. For instance, the papers [29] and [1] address the question as to what kind of geometric information is encoded in them. We refer to Sect. 1 and to [13, Sect. 1] for more on background and motivation.

Even though on surfaces linear series are reasonably well understood thanks to powerful methods such as Reider’s theorem, Seshadri constants are – as Demailly pointed out in [12] – extremely delicate already in the two-dimensional case. For instance, if \( X \) is a generic smooth surface of degree \( d \) in \( \mathbb{P}^3 \), then the Seshadri constants of its hyperplane bundle are unknown when \( d \geq 5 \) (cf. [1] for the case \( d = 4 \)). In light of these facts it seems interesting to study Seshadri constants in the surface case, and to aim for explicit bounds or even explicit values.

Our first result concerns surfaces that come with a fixed embedding into projective space. It is clear that in this case one has \( \varepsilon(L, x) \geq 1 \) at all points \( x \), and it is natural to ask under which circumstances equality holds and which small values bigger than \( 1 \) can occur. Theorem 2.1 answers these questions.

The second result deals with line bundles that are merely assumed to be ample. Work of Ein and Lazarsfeld [14] on surfaces, and Ein-Küchle-Lazarsfeld [13] for the higher dimensional case, shows that there exist universal lower bounds on Seshadri constants if one restricts one’s attention to very general points. Refinements of this type of results are due to Küchle and Steffens [19], Steffens [33] and Xu [35]. On the other hand, well-known examples due to Miranda [21, Proposition 5.12] show that there cannot exist universal lower bounds on Seshadri constants that are valid at arbitrary points. So any bound on Seshadri constants at not necessarily very general points needs to take into account the geometry of the polarized surface \( (X, L) \) in some way. We provide in Theorem 3.1 a bound in terms of the canonical slope of \( L \), an invariant defined in terms of the nef cone of the surface. We also carry out a closer analysis of Miranda’s examples, which illustrates the interplay between this invariant and the Seshadri constant.

We next study Seshadri sub-maximal curves at very general points, i.e. curves causing the Seshadri constant \( \varepsilon(L, x) \) to be below its maximal possible value \( \sqrt{L^2} \) at these points. Naturally, it is of interest to find constraints on