Invariant fields of finite irreducible reflection groups

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Abstract. We prove the following result: If $G$ is a finite irreducible reflection group defined over a base field $k$, then the invariant field of $G$ is purely transcendental over $k$, even if $|G|$ is divisible by the characteristic of $k$. It is well known that in the above situation the invariant ring is in general not a polynomial ring. So the question whether at least the invariant field is purely transcendental (Noether’s problem) is quite natural.

Introduction

A linear group $G$ defined over a field $k$ is called a reflection group if it is generated by elements $\sigma$ with $\text{rank}(\sigma - 1) = 1$. If the order of a finite reflection group is not divisible by the characteristic of $k$, then the invariant ring is isomorphic to a polynomial ring (see, for example, Benson [1, Theorem 7.2.1]). This is no longer true in general if $|G|$ is divisible by $\text{char}(k)$. In fact, the authors classified the finite irreducible reflection groups whose invariant rings are polynomial rings in [7]. In view of this, it is natural to ask if at least the invariant field of a finite reflection group is purely transcendental over $k$. In this paper we show that this is true for all finite irreducible reflection groups. In other words, the answer to Noether’s problem is affirmative for all these groups.

We use the following notation: $V$ is a vector space of finite dimension $n$ over a field $k$ of characteristic $p$ (which may be zero), and $G \leq \text{GL}(V)$ is a finite linear group. Then $G$ also acts on the symmetric algebra $k[V] := S(V)$ of $V$ and on its field of fractions $k(V) := \text{Quot}(k[V])$. We denote the invariant ring and invariant field by $k[V]^G$ and $k(V)^G$, respectively.

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Theorem 1. With the above notation, let $G$ be an irreducible reflection group. Then the invariant field $k(V)^G$ is purely transcendental as an extension of $k$.

Overview of the proof. If $n = \dim(V) = 1$ then $k[V]^G$ is polynomial, and the same is true for $n = 2$ by Nakajima [10, Theorem 5.1]. In fact, the hypothesis that $G$ is an irreducible reflection group is not needed for $n = 2$, since the subfield $k(V)_0^G$ consisting of 0 and all rational invariants which can be written as quotients of homogeneous invariants of equal degrees, has transcendence degree 0 or 1 and hence is purely transcendental by L"uroth's theorem. But $k(V)^G$ is purely transcendental over $k(V)_0^G$ (see Kemper [6, Proposition 1.1]).

If $G$ is imprimitive, then $k[V]^G$ is polynomial by Nakajima [10, Theorem 2.4]. Hence from now on we may assume that $n \geq 3$ and $G$ is primitive.

We use the classification of finite irreducible primitive reflection groups by Kantor, Wagner, Zalesskiš, and Serezhkin. For an overview and references, see Kemper and Malle [7]. From these groups, we only have to consider those whose invariant ring is not polynomial. We get the following list, where not all groups with polynomial invariant rings have been excluded and some groups which are not irreducible primitive reflection groups have been added for simplicity:

1. $\text{SU}_n(q) \leq G \leq \text{GU}_n(q)$, $G \neq \text{SU}_3(2)$, where $p|q$.
2. $\Omega_n^{(\pm)}(q) < G \leq \text{GO}_n^{(\pm)}(q)$, $G \neq \text{SO}_n^{(\pm)}(q)$, where $p|q$, $p \neq 2$.
3. $G = \text{SO}_n^{(\pm)}(q)$, where $p = 2$, $n \geq 4$ is even and $p|q$.
4. $G = \text{Sp}_n(q)$, where $n$ is even and $p|q$.
5. $G = \mathfrak{S}_{n+2}$, where $n \geq 5$, $p|(n + 2)$.
6. $G = W_3(G_{30}, W_3(G_{31}), W_3(G_{32}), W_2(G_{34}), W_3(G_{36}), W_3(G_{37})$, or $W_3(G_{37})$.

Here $\Omega_n^{(\pm)}(q)$ denotes the commutator subgroup of $\text{GO}_n^{(\pm)}(q)$, $\mathfrak{S}_{n+2}$ is the symmetric group of order $(n + 2)!$ with the natural irreducible module of dimension $n$ (see the proof of Proposition 7 for details), and $W_p(G_i)$ is the $p$-modular reduction of the $i$-th complex reflection group in the classification of Shephard and Todd [11].

The cases (1) and (3) are treated in Proposition 3 below. In case (2), $k(V)^G$ is purely transcendental by Kemper [6, Theorem 2.4]. The invariants of symplectic groups have been studied by Carlisle and Kropholler (see Benson [1, Sect. 8.3]). The proof of Theorem 8.3.4 in [loc. cit.] shows that the invariant field is purely transcendental in case (4). Case (5) is treated in Proposition 7 below. Finally, all groups from case (6) are treated in Proposition 8, with the exception of $W_3(G_{37})$, which is treated in Proposition 11.

Most of the proofs are based on a criterion given by the first author in [6, Corollary 1.8], which we repeat in a form which is appropriate for the present purpose.