The Alexander polynomial and finite type 3-manifold invariants

Stavros Garoufalidis · Nathan Habegger

Received: 27 April 1998 / in final form: 8 August 1999

Abstract. Using elementary counting methods, we calculate a universal perturbative invariant (also known as the $LMO$ invariant) of a 3-manifold $M$, satisfying $H_1(M; \mathbb{Z}) = \mathbb{Z}$, in terms of the Alexander polynomial of $M$. We show that $+1$ surgery on a knot in the 3-sphere induces an injective map from finite type invariants of integral homology 3-spheres to finite type invariants of knots. We also show that weight systems of degree $2m$ on knots, obtained by applying finite type $3m$ invariants of integral homology 3-spheres, lie in the algebra of Alexander-Conway weight systems, thus answering the questions raised in [Ga].

1. Introduction

1.1. History

In their fundamental paper, T.T.Q. Le, J. Murakami and T. Ohtsuki [LMO] constructed a map $Z_{LMO}$ which associates to every oriented 3-manifold an element of the graded (completed) Hopf algebra $A(\theta)$ of trivalent graphs.\footnote{For a different construction of $Z_{LMO}(M)$ for a rational homology 3-sphere $M$, see [BGRT2].} The restriction of this map to the set of oriented integral homology 3-spheres was shown in [Le1] to be a universal finite type invariant of integral homology 3-spheres (i.e., every rationally-valued finite type invariant factors through it). Thus $Z_{LMO}$ is a rich (though not fully understood) invariant of integral homology 3-spheres. However, the invariant $Z_{LMO}$ behaves differently as soon as the first Betti number of the 3-manifold, $b_1(M)$, is positive. In [Ha2], the second author used an elementary counting argument to deduce that $Z_{LMO}(M) = 1$, if $b_1(M) > 3$. 

S. GAROUFALIDIS
Department of Mathematics, Brandeis University, Waltham, MA 02254-9110, USA
(e-mail: stavros@oscar.math.brandeis.edu)

N. HABEGGER
UMR 6629 du CNRS, Université de Nantes, Département de Mathématiques,
2 rue de la Houssinière, 44072 NANTES Cedex 03, France
(e-mail: habegger@math.univ-nantes.fr)

The authors were partially supported by NSF grant DMS-95-05105 and by the CNRS respectively. This and related preprints can also be obtained at http://www.math.brown.edu/~stavrosg and at http://www.math.sciences.univ-nantes.fr/preprints/
and to compute $Z^{LMO}(M)$, if $b_1(M) = 3$ (and also for $b_1(M) = 2$, see [BH]), in terms of the Lescop invariant [Les] of $M$. It is an open problem to compute $Z^{LMO}(M)$, for $b_1(M) = 0, 1$.

It is the purpose of the present paper to exploit elementary counting methods in order to calculate $Z^{LMO}(M)$, for 3-manifolds $M$ which satisfy $H_1(M, \mathbb{Z}) = \mathbb{Z}$, in terms of a “classical invariant” of $M$, namely its Alexander polynomial. This includes the special case of 0-surgery$^2$ of a knot $K$ in $S^3$, $S^3_{K,0}$, in which case the Alexander polynomial of $S^3_{K,0}$ is the Alexander-Conway polynomial of $K$. An important ingredient of our computation is the recent result of A. Kricker, B. Spence, and I. Aitchinson, [Kr, KSA], calculating the Conway weight system on uni-trivalent graphs.

Although the invariant $Z^{LMO}(S^3_{K,+1})$, of +1-surgery on a knot $K$ (in contrast to 0-surgery), is not determined by the Alexander-Conway polynomial of $K$ (there are examples with nontrivial invariant, and trivial Alexander polynomial), we show that after truncating $Z^{LMO}$ at degree $m$, the associated degree $2m$ knot weight system lies in the algebra of the Alexander-Conway weight systems. Similar methods allow us to show that finite type invariants of integral homology 3-spheres are determined by their associated knot invariants, thus answering positively the questions (see below) that were posed in [Ga] prior to the construction of the LMO invariant. (At that time, the only known finite type invariant of 3-manifolds was the Casson invariant.)

1.2. Statement of the results

All 3-manifolds and links considered in the present paper will be oriented.

Theorem 1. Let $M$ be a closed, connected 3-manifold satisfying $H_1(M, \mathbb{Z}) = \mathbb{Z}$. The universal invariant $Z^{LMO}(M) \in A(\emptyset)$ can be calculated in terms of the Alexander-Conway polynomial $A(M)$ of the 3-manifold. Conversely, the Alexander polynomial of $M$ can be calculated in terms of the universal invariant $Z^{LMO}(M) \in A(\emptyset)$.

A precise formula relating the two invariants will be given in Section 2.

We outline here the basic idea of the proof, which though somewhat technical, really is quite elementary: If a manifold $M$ is obtained by 0-surgery on a knot $K$ in $S^3$ (the general case of a manifold satisfying $H_1(M, \mathbb{Z}) = \mathbb{Z}$ is not much harder), then quite immediately from the definitions, the degree $m$ part of $Z^{LMO}(M)$ can be computed from the part of the Kontsevich integral of $K$ (written in terms of

---

$^2$ given a framed link $L$ in a 3-manifold $M$, we denote by $M_L$ the result of Dehn surgery on $L$.

$^3$ An earlier version of this paper contained only this special case. We extend special thanks to C. Lescop, for help in extending to the general case and to D. Thurston, for pointing out that the result should hold in this generality.