Constructing infinitely many smooth structures
on $3\mathbb{CP}^2 \# n\mathbb{CP}^2$

B. Doug Park

Received: 17 January 2000 / Published online: 18 January 2002 – © Springer-Verlag 2002

Abstract. Using Seiberg-Witten theory and rational blow-down procedures of R. Fintushel and R.J. Stern, we construct infinitely many irreducible smooth structures, both symplectic and non-symplectic, on the four-manifold $3\mathbb{CP}^2 \# n\mathbb{CP}^2$ for each integer $n$ lying in the interval $10 \leq n \leq 13$.

Mathematics Subject Classification (2000): 57R55, 57R57, 53C15, 57M60

1. Introduction

This paper is a continuation of the study initiated in [P1] and [P2]. For some history and general remarks on distinguishing smooth structures on $3\mathbb{CP}^2 \# n\mathbb{CP}^2$, we refer the reader to Section 1 of [P1]. The author’s previous results can be summarized in the following

Theorem 1. (cf. [P1], [P2]) There exists a smooth closed simply-connected irreducible symplectic 4-manifold $X_n$ that is homeomorphic but not diffeomorphic to $3\mathbb{CP}^2 \# n\mathbb{CP}^2$ for each integer $10 \leq n \leq 13$.

Recall that a smooth 4-manifold $X$ is said to be irreducible if it possesses no non-trivial connected sum decomposition, i.e.

$$X = M \# N \implies M \text{ or } N \text{ is a homotopy } S^4$$

whenever $M$ and $N$ are smooth 4-manifolds. To be more precise, we should say that the particular smooth structure in question on the topological manifold $X$ is irreducible. Our main new result is the following

Theorem 2. There are infinitely many irreducible smooth structures, both symplectic and non-symplectic, on the 4-manifold $3\mathbb{CP}^2 \# n\mathbb{CP}^2$ for each integer $10 \leq n \leq 13$.

We remark that, as of this writing, the examples in the above theorems have the smallest Euler characteristic among all closed orientable simply-connected
4-manifolds with $b^+_2 > 1$ that are known to possess more than one smooth structure. Also note that by blowing up our examples repeatedly, we obtain infinitely many (albeit reducible) symplectic and non-symplectic smooth structures on $3\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ for all $n \geq 10$.

2. General construction

First we review the construction of $X_n$. For each $n$, we start out with two symplectic 4-manifolds $M_1$ and $M_2$. We locate 2-dimensional symplectic submanifolds $\Sigma_j \hookrightarrow M_j$ ($j = 1, 2$) each having genus $g(\Sigma_j) = 2$, and self-intersection $[\Sigma_j]^2 = 0$. We take out the tubular neighborhoods of $\Sigma_j$ and get manifolds with boundary $M_j - (D^2 \times \Sigma_j)$. Glue these manifolds together using a suitably chosen orientation-reversing diffeomorphism $\psi$ between the boundaries $S^1 \times \Sigma_j$. We define

$$X_n = [M_1 - (D^2 \times \Sigma_1)] \cup_\psi [M_2 - (D^2 \times \Sigma_2)].$$

It is well-known that the resulting sum $X_n$ can be given a canonical symplectic structure by patching together the symplectic forms on $M_j$ (cf. [Go]). For the specific combinations of $M_j$ that were used in the construction, we refer to Table 1.

<table>
<thead>
<tr>
<th>$X_{10}$</th>
<th>$X_{11}$</th>
<th>$X_{12}$</th>
<th>$X_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>$W$</td>
<td>$W$</td>
<td>$T^4 # 2\overline{\mathbb{CP}}^2$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$S$</td>
<td>$E(1)_K$</td>
<td>$S$</td>
</tr>
</tbody>
</table>

Here, $W$ is the double of the Kodaira-Thurston manifold. (This is the manifold $Q_2$ in [Go].) $S$ is the numerical Godeaux surface discovered by Craigero and Gattazzo (cf. [CG], [DW]). $E(1)_K$ denotes the homotopy rational elliptic surface of Fintushel and Stern, where $K$ is the trefoil knot (cf. [FS2]).

The goal of this section is to construct a collection of smooth 4-manifolds, $X_n(p)$, given any integer $p > 1$. We assume that there is a smoothly embedded torus $f \subset X_n$ satisfying the following three conditions:

(i) The homology class $[f] \in H_2(X_n; \mathbb{Z})$ is not torsion (in particular we require $[f] \neq 0$), and has self-intersection $[f]^2 = 0$.

(ii) $[f]$ is also represented by an immersed 2-sphere $\sigma$ having a unique positive double point. (The tubular neighborhood of such a torus is called a “fishtail” neighborhood.)

(iii) $\pi_1(X_n - f) = 1$. 