\section{Introduction}

Let \((X^n, \omega)\) be a complete, Kähler manifold of complex dimension \(n\) (we will sometimes abuse notation and let \(\omega\) denote both the Kähler form and the metric induced by the Kähler form). The metric allows one to define the class of square-integrable forms of all bi-degrees, \(\Omega^{p,q}_g(X)\). The metric also defines an adjoint, \(d^\ast\), to exterior differentiation and thus Laplace operators, \(\Delta\), at all form levels. Let \(H^{p,q}_g(X)\) denote those forms \(\phi \in \Omega^{p,q}_g(X)\) which are harmonic: \(\Delta \phi = 0\) in, say, the weak sense. The condition of square-integrability is strong and the size of \(H^{p,q}_g(X)\) is small compared to the space of all harmonic forms; in many situations, e.g., \((X, \omega) = \text{hyperbolic upper half plane in } \mathbb{C}^n\), it happens that \(H^{p,q}_g(X) = 0\) unless \(p + q = n\). We mention that the middle dimension, when \(p + q = n\), is always a special case and there are no results about \(L^2\) harmonic forms in these dimensions in this paper.

The main purpose of this paper is to prove two vanishing results on \(H^{p,q}_g(X)\) when \(p + q \neq n\), under a growth assumption on a primitive of \(\omega\). Assume \(\omega\) is given by a global potential function, i.e., there is a \(\lambda \in C^2(X)\) such that \(\omega = i \partial \bar{\partial} \lambda\). The crucial condition is a pointwise comparison of \(\partial \lambda\), measured in the metric \(\omega\), and \(\lambda\) itself (see Definition 1 below). Our results (see Theorems 2.1 and 2.6) are: (1) if the gradient of \(\lambda\) is dominated by a constant times \(\lambda\), then \(H^{p,q}_g(X) = 0\) if \(p + q \neq n\), and, (2) if the gradient of \(\lambda\) is strictly less than \(\lambda\), then, in addition to \(H^{p,q}_g(X) = 0\), we obtain a lower bound on \((\Delta \phi, \phi)\) for \(\phi \in \Omega^{p,q}_g(X)\), \(p + q \neq n\).

A secondary purpose of the paper is the estimation of two \(L^2\) extreme value problems. These problems are related to the domination conditions above, if \(\omega\) is the Bergman metric, but are also of independent interest.
There are several previous results directly related to our vanishing results. We perhaps should first point out that, if $X$ is compact, there are the classical vanishing theorems of Bochner-Hodge-Kodaira-et.al., which apply with various hypotheses on $\omega$ and/or on bundles over $X$, see [Gr-Ha]. The first general result in the non-compact case is due to Donnelly and Fefferman, [Don-Fef]. If $X$ is a strongly pseudoconvex domain in $\mathbb{C}^n$, they showed in [Don-Fef] that $\mathcal{H}^{p,q}_{(2)}(X) = 0$, $p + q \neq n$, if $\omega$ is the Bergman metric. Their proof used the asymptotic expansion of the Bergman kernel known to hold in this case [Fef] (through difficult analysis), but completely unavailable in general. They also showed that $\dim \mathcal{H}^{p,q}_{(2)}(X) = \infty$ if $p + q = n$. In a more abstract direction, Gromov, [Gro], introduced the notion of Kähler hyperbolicity (see Remark 2 at the end of Sect. 1) and established the vanishing of $\mathcal{H}^{p,q}_{(2)}(X)$, outside the middle dimension, for any $(X, \omega)$ which is Kähler hyperbolic and which covers a compact manifold. Gromov also showed, for such $X$, that $\dim \mathcal{H}^{p,q}_{(2)}(X) = \infty$, when $p + q = n$. Then, in two papers, [Don1] and [Don2], Donnelly re-examined the case of the Bergman metric. In [Don1], he showed how ideas from [Gro] allowed one to obtain the vanishing result on strongly pseudoconvex domains without using the asymptotic expansion of the Bergman kernel. Most interestingly for our purposes, in [Don2], Donnelly showed two things: (a) the vanishing theorem for the Bergman metric extends to certain weakly pseudoconvex domains, and (b) there are weakly pseudoconvex domains which are not Kähler hyperbolic with respect to the Bergman metric. We also point out the work of Herbort [Her] and Tiao [Tia] as related to our estimation of the extreme value problems.

Finally, after work on this paper was completed, we obtained the preprints [C-X] and [J-Z]. In both of these papers, the authors prove that $\mathcal{H}^{p,q}_{(2)}(X) = 0$, $p + q \neq n$, if $\omega = d\alpha$ with $|\alpha|_{\omega}$ growing slower than the Riemannian distance associated to $\omega$; the proof of this result (in both papers) is quite similar to the proof of our Theorem 2.6.

Our interest in the vanishing of $\mathcal{H}^{p,q}_{(2)}(X)$ is analytical: we are interested in solving the Cauchy-Riemann equations, $\bar{\partial}u = \alpha$, with estimates, and to do so requires that the data $\alpha$ satisfies $\alpha \perp \mathrm{Ker} \bar{\partial}^*$. This is equivalent to $\bar{\partial}^* \alpha = 0$ and $H\alpha = 0$, where $H : \Omega^{p,q}_{(2)}(X) \rightarrow \mathcal{H}^{p,q}_{(2)}(X)$ is orthogonal projection, and so the size of $\mathcal{H}^{p,q}_{(2)}(X)$ is important. We mention, however, that there are topological reasons to study $L^2$ cohomology, among them the so-called Hopf-Chern-Singer conjecture about the sign of the Euler characteristic of a negatively curved $X$, and there is a very extensive literature on $L^2$ cohomology. See [Gro], [Dod], [Lot], and their references for an introduction to this body of work.

We also mention that Gromov’s theorem amounts to imposing a condition on solvability of the $d$-equations with $L^\infty$ control (for the special right hand side $\omega$),

1 In both (a) and (b), the domains are of finite type, a somewhat tractable class of domains, see [D’A]. For the positive results (a), the estimates on the Bergman kernel in [Cat2] and [Mc3] are used.