Nonrelativistic limit in the energy space for nonlinear Klein-Gordon equations

Shuji Machihara · Kenji Nakanishi · Tohru Ozawa

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Abstract. We study the nonrelativistic limit of the Cauchy problem for the nonlinear Klein-Gordon equation and prove that any finite energy solution converges to the corresponding solution of the nonlinear Schrödinger equation in the energy space, after the infinite oscillation in time is removed. We also derive the optimal rate of convergence in $L^2$.

1. Introduction

In this paper we study the nonrelativistic limit of the Cauchy problem for the nonlinear Klein-Gordon equation:

\[
\frac{\hbar}{2mc^2} \ddot{u} - \frac{\hbar}{2m} \Delta u + \frac{mc^2}{2} u + f(u) = 0,
\]

where $u = u(t, x): \mathbb{R}^{1+n} \rightarrow \mathbb{C}$, $f(u) = \lambda |u|^p u$ with $p > 0$ and $\lambda \in \mathbb{R}$, $c$ is the speed of light, $\hbar$ is the Planck constant, and $m > 0$ is the mass of particle. We consider the pure power nonlinearity just for simplicity. Our arguments below are obviously applicable to more general functions $f(\cdot)$.

Rescaling $t, x, u, \lambda$ and $c$, we can normalize the other constants as $\hbar = m = 2$. Before taking the nonrelativistic limit $c \to \infty$, we consider the modulated function $v := e^{-ic^2 t} u$, which obeys the following modulated equation:

\[
\ddot{v}/c^2 + 2i \dot{v} - \Delta v + f(v) = 0.
\]

Then we may think of the nonlinear Schrödinger equation:

\[
2i \dot{v} - \Delta v + f(v) = 0
\]
as the singular limit as $c \to \infty$ of (1.2). The most important quantities of these equations are the energy and the charge:

$$E_{NK}(u) := \int_{\mathbb{R}^n} \frac{1}{c} |\dot{u}|^2 + |\nabla u|^2 + F(u) \, dx,$$

$$E_{NS}(u) := \int_{\mathbb{R}^n} |\nabla u|^2 + F(u) \, dx,$$

$$Q_K(u) := \int_{\mathbb{R}^n} |u|^2 + \Im \frac{\dot{u}}{c} \bar{u}^2 \, dx,$$

$$Q_S(u) := \int_{\mathbb{R}^n} |u|^2 \, dx,$$

(1.4)

where $F(u) := 2\lambda |u|^p + 2/(p + 2)$. $E_{NK}$ and $Q_K$ are conserved for any solution $u$ of (1.2), and so are $E_{NS}$ and $Q_S$ for (1.3).

So the most natural question about the nonrelativistic limit of (1.2) is whether any solution with finite energy and charge converges to a solution of (1.3) in the topology induced by those quantities ($H^1$). However, to the best of our knowledge, there is no rigorous result on this problem in the literature. Indeed, there are a few papers on some weaker results. In [11], $L^2$ convergence was proved assuming $H^2$ convergence of the initial data in the case where $n \leq 2$ and $p \leq 2$. In [9, 8], $L^q$ convergence was shown for $2 \leq q < 2n/(n - 2)$ assuming $H^1$ boundedness and $L^2$ convergence of the initial data under the assumption $n \leq 3$ and some restrictive assumptions on $p$.

Here we present an almost complete answer. Namely, we will prove $H^1$ convergence assuming $H^1$ convergence of the initial data, for any $n$ and any $p < 4/(n - 2)$. Perhaps the upper bound $p = 4/(n - 2)$ might be allowed, but it would be a very delicate and difficult problem. Actually, we do not know even the uniqueness of finite energy solutions for (1.3) with $p = 4/(n - 2)$ in general case (see [2] for the wellposedness in the radial case).

As we will show later, the $H^1$ convergence can be rather easily shown, at least when $\lambda \geq 0$, by a compactness argument. However, when we want to investigate the nonrelativistic limit in more details, such an argument can yield very little information. So we give another method of analyzing the problem via the Strichartz estimate applied to the associated integral equation.

Here the main idea is to adjust the space-time norms to the nonrelativistic limit in order to get a uniform estimate. More precisely, we split the solution in the Fourier space into the lower frequency part $|\xi| < c$ and the higher frequency part $|\xi| > c$. The lower part is shown to behave as a solution of the Schrödinger, and the higher part vanishes at the rate of a certain power of $c$. Then one might be anxious about the compatibility of such a decomposition with the nonlinearity, but it will be efficiently dealt with the nonlinear estimate in sum spaces of Lebesgue-Besov type. At this step, we lose the information about the frequency separation, but we can recover it by exploiting the $\nabla/c$-derivative gain.