Quantitative estimates for periodic points of reversible and symplectic holomorphic mappings

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Abstract. We show that reversible holomorphic mappings of \( \mathbb{C}^2 \) have periodic points accumulating at an elliptic fixed point of general type. On the contrary, we also show the existence of holomorphic symplectic mappings that have no periodic points of certain periods in a sequence of deleted balls about an elliptic fixed point of general type. The radii of the balls are carefully chosen in terms of the periods, which allows us to show the existence of holomorphic mappings of \( \mathbb{C}^2 \) that are not reversible with respect to any \( \mathcal{C}^1 \) involution with a holomorphic linear part, and that admit no invariant totally real and \( \mathcal{C}^1 \) real surfaces.

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1 Introduction

Let \( \varphi \) be a biholomorphic mapping defined near \( 0 \in \mathbb{C}^2 \) with \( \varphi(0) = 0 \). The origin is an elliptic fixed point of \( \varphi \) of general type, if in some local holomorphic coordinates \( \varphi \) is of the form

\[
\begin{align*}
\xi_1 &= \lambda \xi e^{a_1(\xi \eta)^s} + o(2s^2 + 1), \\
\eta_1 &= \lambda \eta e^{-a_1(\xi \eta)^s} + o(2s^2 + 1),
\end{align*}
\]

in which \( (\xi, \eta) \) are the holomorphic coordinates of \( \mathbb{C}^2 \). One says that \( \varphi \) is symplectic if \( \varphi^* d\xi \wedge d\eta = d\xi \wedge d\eta \), and that it is reversible near the origin with respect to an involution \( \tau (\tau^2 = \text{Id}, \tau(0) = 0) \) if \( \varphi^{-1} = \tau \varphi \tau^{-1} \). The main purpose of this paper is to study the existence of periodic points near fixed points of the holomorphic mappings that are either symplectic or reversible.

The existence of periodic points of area-preserving or reversible mappings of the real plane was established by G. D. Birkhoff, who showed that the mappings have periodic points accumulating at each elliptic fixed point of general type.
The periodic points of reversible real mappings and systems in higher dimensional spaces were further studied by R. L. Devaney [3]. To motivate our results, let us recall Birkhoff’s treatment for real planar mappings. Start with a twist mapping of the $z$-plane

$$T : z_1 = \lambda z e^{i(c z)}, \quad i = \sqrt{-1}, \quad |\lambda| = 1.$$  

Away from the origin the fixed points of the $n$-th iterate $T^n$ form circles surrounding the origin. Such circles of periodic points can, of course, be destroyed by perturbing $T$. However, these circles do survive in a weaker sense: If $S$ is a higher order (smooth or real analytic) perturbation of $T$, each circle of periodic points of $T$ is deformed slightly into a closed curve $C$, surrounding the origin, such that $S^n$ sends each point on $C$ to a point in the radial direction. We shall call such a curve $C$ the Birkhoff curve of $S$ of order $n$. Birkhoff then went on to find periodic points as follows: If $S$ is additionally area-preserving, the intersection of the Birkhoff curve $C$ with $S^n(C)$ is evidently non-empty and consists of fixed points of $S^n$ (see [2]); if $S$ is additionally reversible with respect to an involution $\tau$ and if the set of fixed points of $\tau$ is real line, then $C$ intersects the curve of fixed points of $\tau$ and the intersection consists of periodic points of $S$ with period dividing $2n$ (see [1]).

Returning to the holomorphic case, we first want to show that certain Birkhoff (holomorphic) curves still exist for a holomorphic mapping $\varphi$ of the form (1.1), and that the Birkhoff curves, as in the real case, yield periodic points when $\varphi$ is reversible. However, the Birkhoff curves behave quite differently in case of holomorphic symplectic mappings. The main results of this paper conclude that for a Birkhoff curve of $C$ of order $n$, $\varphi^n(C)$ might not intersect with $C$ in a certain neighborhood of the origin; see also Sect. 5 for a holomorphic symplectic mapping $\varphi$ which has a Birkhoff curve $C$ of order 1 with $\varphi(C) \cap C = \emptyset$.

To formulate our results, we need some notation. For a complex number $\lambda$ with $|\lambda| = 1$, put

\begin{equation}
(1.2) \quad \kappa_n(\lambda) = \begin{cases} 
\left[\frac{|\lambda^n - 1|}{n}\right]^\frac{1}{n} & \text{if } \lambda^n \neq 1, \\
\left[\frac{1}{n}\right]^\frac{1}{n} & \text{otherwise.}
\end{cases}
\end{equation}

Let $B(r) \subset \mathbb{C}^2$ be the ball centered at the origin with radius $r$, and let $B^*(r) = B(r) \setminus \{0\}$. Recall that $p$ is a period point of $\varphi$ if $\varphi^n(p) = p$ for some positive integer $n$, and the smallest such positive integer $n$ is called the period of the periodic point.

**Theorem 1.1.** Let $\varphi$ be a homeomorphism defined near the origin of $\mathbb{C}^2$ by (1.1) with $\lambda^2 \neq 1$. Suppose that $\varphi$ is reversible with respect to a $C^1$ involution of which

(1.1) \quad \varphi(z) = \lambda z e^{i(c z)}, \quad i = \sqrt{-1}, \quad |\lambda| = 1.