Rigidity of group actions on solvable Lie groups

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Received: 27 August 1999

Abstract. We establish analogs of the three Bieberbach theorems for a lattice \( \Gamma \) in a semidirect product \( S \rtimes K \) where \( S \) is a connected, simply connected solvable Lie group and \( K \) is a compact subgroup of its automorphism group. We first prove that the action of \( \Gamma \) on \( S \) is metrically equivalent to an action of \( \Gamma \) on a supersolvable Lie group. The latter is shown to be determined by \( \Gamma \) itself up to an affine diffeomorphism. Then we characterize these lattices algebraically as polycrystallographic groups. Furthermore, we realize any polycrystallographic group \( \Gamma \) as a lattice in a semidirect product \( S \rtimes F \) with \( F \) being a finite group whose order is bounded by a constant only depending on the dimension of \( S \). This generalization of the first Bieberbach theorem is used to obtain a partial generalization of the third one as well. Finally we show for any torsion free closed subgroup \( \Gamma' \subset S \rtimes K \) that the quotient \( S/\Gamma' \) is the total space of a vector bundle over a compact manifold \( B \), where \( B \) is the quotient of a solvable Lie group by a torsion free polycrystallographic group.

1 Introduction and main results

The classical Bieberbach theorems investigate the structure of crystallographic groups, i.e. of discrete cocompact subgroups of the isometry group of the Euclidean space \( \text{Iso}(\mathbb{R}^d) = \mathbb{R}^d \rtimes O(d) \).

Bieberbach’s First Theorem. Let \( \Gamma \subset \mathbb{R}^d \rtimes O(d) \) be a crystallographic group. Then \( \Gamma \cap \mathbb{R}^d \) has finite index in \( \Gamma \).

Bieberbach’s Second Theorem. Let \( \Gamma_1, \Gamma_2 \subset \mathbb{R}^d \rtimes O(d) \) be two crystallographic groups. Suppose there exists an isomorphism \( \iota: \Gamma_1 \to \Gamma_2 \) of abstract groups. Then \( \iota \) is given by conjugation with an element in the group of affine motions \( \mathbb{R}^d \rtimes \text{GL}(d) \).

Bieberbach’s Third Theorem. In each dimension there are only finitely many isomorphism classes of crystallographic groups.

We will study discrete, cocompact subgroups of semidirect products \( S \rtimes K \) where \( S \) is a connected, simply connected solvable Lie group and \( K \) is a compact
subgroup of its automorphism group $\text{Aut}(S)$. Recall that a connected solvable Lie group $S$ contains closed subgroups
\[{e} = N_1 \subset \cdots \subset N_k = S\]
such that $N_i$ is normal in $N_{i+1}$ and $N_{i+1}/N_i \cong \mathbb{R}$ or $N_{i+1}/N_i \cong S^1$. If the groups $N_1, \ldots, N_k$ are normal in $S$, the group $S$ is called supersolvable. A connected nilpotent Lie group is supersolvable; the converse however is not true.

The automorphism group $\text{Aut}(S)$ of a simply connected Lie group $S$ is a Lie group with finitely many connected components. Consequently, any compact subgroup of $\text{Aut}(S)$ is contained in a maximal compact subgroup and all maximal compact subgroups are conjugate, compare Remark 3.1. Notice that a semidirect product $S \rtimes K$ acts on $S$ by $(\tau, A) \cdot v = \tau \cdot A(v)$ for $(\tau, A) \in S \rtimes K$, $v \in S$.

**Theorem 1.** Let $S$ be a connected, simply connected solvable Lie group, and let $K \subset \text{Aut}(S)$ be a maximal compact subgroup. Then there is

a) a unique maximal connected, simply connected supersolvable normal subgroup $R$ of $S \rtimes K$ such that $K$ also can be viewed as a subgroup of $\text{Aut}(R)$,

b) an isomorphism $\iota: R \rtimes K \to S \rtimes K$ of Lie groups and

c) an equivariant isometry $f: (R, g_1) \to (S, g)$ for suitable left invariant metrics $g_1, g$ on $R$ and $S$, i.e. $f(h \star v) = \iota(h) \star f(v)$ for $v \in R$ and $h \in R \rtimes K$.

More precisely, if $g$ is a left invariant metric on $S$ such that $g|_{e}$ is invariant under the natural representation of $K$ in the Lie algebra $s$ of $S$, then the pull back metric $g_1 := f^*g$ is left invariant on $R$.

So we may restrict attention to actions on supersolvable Lie groups, and thereby we can view the following theorem as an analogue of the second Bieberbach theorem:

**Theorem 2.** Let $S_i$ be a connected, simply connected supersolvable Lie group, $K_i \subset \text{Aut}(S_i)$ a compact subgroup, and let $\Gamma_i \subset S_i \rtimes K_i$ be a discrete cocompact subgroup, $i = 1, 2$. Suppose there exists an isomorphism $\iota_i: \Gamma_1 \to \Gamma_2$ of abstract groups. Then there is an isomorphism $\varphi: S_1 \to S_2$ and an element $\tau \in S_2$ such that the affine diffeomorphism
\[f: S_1 \to S_2, \quad v \mapsto \varphi(v) \cdot \tau\]
is equivariant. In particular, $\iota_i(\gamma) \star w = f(\gamma \star f^{-1}(w))$ for all $w \in S_2$, $\gamma \in \Gamma_i$.

In the special case of a nilpotent Lie group $S$ the theorem is due to Auslander (1961a), and the group $\Gamma_i$ is then called an almost crystallographic group.

Theorem 2 is also a partial generalization of the main result of Farrell and Jones (1997). They proved for a pair of torsion free, closed, cocompact subgroups $\gamma_i \subset S_i \rtimes K_i$ for which the identity components are contained in the nilradicals of $S_i$ ($i = 1, 2$) that the following holds: Any isomorphism $\pi_1(S_1/\gamma_1) \to$