\textbf{L}^p - \textbf{L}^q \text{ estimates of solutions to the damped wave equation in 3-dimensional space and their application}

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Abstract. It has been asserted that the damped wave equation has the diffusive structure as \( t \to \infty \). In this paper we consider the Cauchy problem in 3-dimensional space for the linear damped wave equation and the corresponding parabolic equation, and obtain the \( L^p - L^q \) estimates of the difference of each solution, which represent the assertion precisely. Explicit formulas of the solutions are analyzed for the proof. The second aim is to apply the \( L^p - L^q \) estimates to the semilinear damped wave equation with power nonlinearity. If the power is larger than the Fujita exponent, then the time global existence of small weak solution is proved and its optimal decay order is obtained.

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1. Introduction

Consider the Cauchy problem for the damped wave equation in 3-dimensional space

\[
\begin{aligned}
V_{tt} - \Delta V + V_t &= 0 & \text{on} & & \mathbb{R}^3 \times \mathbb{R}_+, & \mathbb{R}_+ = (0, \infty) \\
(V, V_t)|_{t=0} &= (V_0, V_1)(x), & x &= (x_1, x_2, x_3) \in \mathbb{R}^3,
\end{aligned}
\]  

where

\[ \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \]

It has been indicated by several mathematicians (Li [15], Bellout and Friedman [1], etc.) that the damped wave equation has the diffusive structure as \( t \to \infty \). Our aims in this paper are both to represent the assertion precisely and to apply it to the nonlinear problem for the damped wave equation.

Our first aim is represented in the following theorem.
Theorem 1.1. Let \((V_0, V_1)\) be in \(L^q\) for \(q \geq 1\). Then the solution \(V\) to (1.1) in the distributional sense satisfies the \(L^p - L^q\) estimate

\[
\|(V - \phi)(\cdot, t) - e^{-t/2}W(t; V_0, V_1)\|_{L^p} \leq Ct^{-\frac{3}{2}(1 - \frac{1}{p})^{-1}} \|V_0, V_1\|_{L^q} \tag{1.2}
\]

for \(t \geq t_0 > 0\) with \(1 \leq q \leq p \leq \infty\), where \(t_0\) is arbitrarily fixed time, \(C\) is a positive constant, \(\phi\) is the solution to the Cauchy problem for the corresponding parabolic equation

\[
\begin{align*}
\phi_t - \Delta \phi &= 0, & & \text{on } \mathbb{R}^3 \times \mathbb{R}_+ \\
\phi|_{t=0} &= (V_0 + V_1)(x),
\end{align*} \tag{1.3}
\]

and, by denoting \(v := W(t)g\) by the solution in the distributional sense to the Cauchy problem for the wave equation

\[
\begin{align*}
v_{tt} - \Delta v &= 0, & & \text{on } \mathbb{R}^3 \times \mathbb{R}_+ \\
(v, v_t)|_{t=0} &= (0, g)(x),
\end{align*} \tag{1.4}
\]

\[
W_0(t; V_0, V_1) = (\frac{1}{2} + \frac{t}{8})W(t)V_0 + \partial_t(W(t)V_0) + W(t)V_1. \tag{1.5}
\]

Remark. By the definition of \(W(t)g\), \(e^{-t/2}W_0(t; V_0, V_1)|_{t=0} = V_0\) and \(\partial_t(e^{-t/2}W_0(t; V_0, V_1))|_{t=0} = e^{-t/2}(-\frac{1}{2}W(t)V_0) + \partial_tW(t; V_0, V_1)|_{t=0} = V_1.\)

It is well-known that the solution \(\phi\) to (1.3) on \(\mathbb{R}^N \times \mathbb{R}_+\) satisfies

\[
\|\partial^\alpha_x \partial^\beta_t \phi(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2} - \beta} \|V_0 + V_1\|_{L^q(\mathbb{R}^N)} \tag{1.6}
\]

for \(1 \leq q \leq p \leq \infty\) (see e.g. Ponce [25]), and that the solution \(V\) to (1.1) on \(\mathbb{R}^N \times \mathbb{R}_+\) satisfies

\[
\|\partial^\alpha_x \partial^\beta_t V(\cdot, t)\|_{L^p(\mathbb{R}^N)} \leq C(1 + t)^{-\frac{N}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{1}{2} - \beta}(\|V_0, V_1\|_{L^q(\mathbb{R}^N)}) + \|V_0\|_{H^\frac{N}{2}(\mathbb{R}^N)} + \|V_1\|_{H^\frac{N}{2}(\mathbb{R}^N)} \tag{1.7}
\]

for \(1 \leq q \leq 2 \leq p \leq \infty\) (see Matsumura [18]). Here and after \(c, c_k, C, C_k\) etc. denote generic constants. We note that both \(p \geq 2\) and the regularity of the initial data are necessary in (1.7). Therefore, compared to them, Theorem 1.1 implies the properties of the damped wave equation, which are summarized as follows.

(i) The damped wave equation has not the smoothing effect, different from the parabolic equation. If there is a singularity, then it propagates along the light cone, which is the wave property, though its strength decays exponentially, which is from the damping effect.

(ii) If the singularities are removed, then the asymptotic profile of the solution to the damped wave equation is that of the corresponding parabolic equation. If the initial data have sufficient regularity, then the singularity is, in fact, removed, so that the damped wave equation may have the same properties as those to the parabolic equation under the suitable situations.