Tight closure and linkage classes in Gorenstein rings

Adela Vraciu

Department of Mathematics, University of Kansas, Lawrence, KS 66045, USA

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Abstract. We study the relationship between the tight closure of an ideal and the sum of all ideals in its linkage class

1. Introduction

Tight closure and linkage have been developed as independent branches of commutative algebra. Tight closure was introduced by Hochster and Huneke in [HH1] as a tool for studying ideals in rings of positive characteristic, while the theory of linkage (liaison) has its roots in the study of curves in three dimensional projective space ([Ap], [Ga], [R], etc.). The algebraic foundations of linkage were established in [PS2], [AN], [HU], et cetera.

The main result of this paper (Thm. 2) establishes a connection between these two theories. In the process of proving Thm. 2 we establish a relationship between ideals \( J \) in the linkage class of an ideal \( I \) and ideals in the linkage class of an \( \mathfrak{m} \)-primary ideal \( (I, x_1, \ldots, x_{d-g}) \), where \( d = \dim(R) \), \( g = \text{ht}(I) \), and the choice of \( x_1, \ldots, x_{d-g} \) depends on \( J \) (Prop. 6). This result might be of independent interest. We also develop a theory of corner powers of unmixed ideals, which are obtained as direct links of Frobenius powers. We explore some of their properties in Section 3, and we use them as a tool in the proof of Thm. 2.

The setting is that of a Gorenstein ring of positive characteristic, where one can use the information available about the tight closure of ideals of finite projective dimension to relate the tight closure of an unmixed ideal to ideals in its linkage class.

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The investigation carried out in this paper is motivated by the following question: If $R$ is a characteristic $p$ ring with test ideal $\tau$, and $I$ is an arbitrary ideal, what is the relationship between $I : \tau$ and $I^*$? The definition of the test ideal (see 2) implies that $I : \tau \supset I^*$, and in the particular case of a Gorenstein characteristic $p$ ring $R$ and an ideal $I$ of finite projective dimension we actually have equality: $I^* = I : \tau$ (see 1).

This equality is far from true in general. We propose to seek ideals that multiply $I : \tau$ into $I^*$. Note that in the case when $I : \tau$ is multiplied into $I^*$ by an $m$-primary ideal, it follows that $I$ admits a test exponent, and therefore the tight closure of $I$ commutes with localization (see [HH2] for details about test exponents and localization of tight closure).

The main result of this paper (Thm. 2) states the following: if $R$ Gorenstein and $I$ is unmixed, then $(\bar{I})^*(I : \tau) \subset I^*$, where $\bar{I}$ denotes the sum of all the ideals in the linkage class of $I$. As an application, it follows that tight closure commutes with localization for any ideal $I$ for which $\bar{I}$ is $m$-primary (Cor. 4).

We do not expect that $\bar{I}$ is the largest ideal with this property, but this result is interesting in light of the unexpected relationship between tight closure and linkage. There are several interesting consequences pertaining to properties of the linkage class of certain ideals. For instance, if $I$ is an unmixed tightly closed ideal containing the test ideal, then $I$ is maximal in its linkage class, in the sense of containing every ideal in its linkage class (see Cor. 2). This provides a class of examples addressing the question raised in [PU]: for which ideals $I$ is every ideal in the linkage class of $I$ contained in $I$?

2. Preliminaries

In this paper, $(R, m)$ denotes a Gorenstein local ring of characteristic $p > 0$ and $q = p^e$ denotes a power of $p$. By parameter ideal we mean an ideal generated by part of a system of parameters.

We recall the relevant definitions:

**Definition 1.** Let $I$ be an ideal of $R$. For every $q = p^e$, $I^{[q]} := (i^q \mid i \in I)$ is called the Frobenius $q$th power of $I$. $R^0$ denotes the set of elements in $R$ which are not in any minimal prime.

An element $x \in R$ is in the tight closure of $I$ if there exists $c \in R^0$ such that $cx^q \in I^{[q]}$ for all $q = p^e$. We write $x \in I^*$.

**Definition 2.** The test ideal of $R$ is the ideal $\tau$ generated by all elements $c \in R^0$ such that for every ideal $I$ and every $x \in I^*$ we have $cx \in I$.

**Definition 3.** Let $(R, m)$ be a Noetherian local ring. An ideal $I$ of height $g$ is called unmixed if all the associated prime ideals of $I$ have height $g$.

**Definition 4.** An ideal $a$ is called Gorenstein if the ring $R/a$ is Gorenstein.

Note that if $R$ is a Gorenstein ring, then any parameter ideal is a Gorenstein ideal. If $R$ is Gorenstein and $R/a$ has finite projective dimension over $R$, $R/a$ is Gorenstein if and only if its minimal free resolution over $R$ is self-dual.

The following property of unmixed ideals is well-known: